Roebuck and Company, and on Thursday afternoon the new Field Museum and the Newberry Library. The visiting ladies were pleasantly entertained on Thursday afternoon at Ida Noyes Hall by the director, Mrs. Goodspeed. Ample cafeteria service was afforded on the campus, and many small groups lunched each day at the Quadrangle Club.

On Wednesday evening the large number of 167 persons shared in the joint dinner of the Society, the Association, Section A of the American Association, and the American Astronomical Society. Professor D. E. SMITH acted as toastmaster, and brief speeches were made by Professor G. A. BLISS for the Society, Professor PHILIP FOX for the Astronomical Society, Professor G. A. MILLER for the Association, Professor D. R. CURTISS for Section A, Professor FLORIAN CAJORI for the newly organized Section L, and Mr. VINCENT W. BROWN, a representative of the Chamber of Commerce of St. Louis, who spoke of the necessity of a thorough training in the fundamentals and of a sound basis of mathematics for practically all lines of business. Professor C. I. PALMER exhibited a copy of the 1637 Descartes geometry, which Professor Cajori in his address of Wednesday had said he had never seen; this copy came to Professor Palmer from earlier possession by Professors Sylvester and George Bruce Halsted. At the close of the speeches Professor E. H. Moore read a reply from Professor Oskar Bolza, expressing appreciation of a joint greeting sent to him by those who attended the mathematical meetings last September.

It is gratifying to note here that the Association, as also the Society, has been invited to affiliate with the American Association for the Advancement of Science, and on this basis the Secretary-Treasurer took part in that association's Council meeting on Thursday morning, at which Professor E. H. MOORE was elected president of the American Association for the next year.

On motion of Professor Rietz at the final session, the Association adopted by a rising vote a resolution expressing appreciation of the courtesy and cordial welcome extended by the department of mathematics of the University of Chicago and of the work of the local committee that contributed so much to the very successful meetings.

The program for the various sessions continued the previous practice of the Association in including expository papers of a fairly elementary character and papers of historical interest. The latter was especially fitting at this time when Section L (Historical and Philological Sciences) of the American Association for the Advancement of Science was being organized. Professor Cajori presided at the Tuesday morning joint session with Section L, Professor Curtiss, retiring vice-president of Section A, at the Wednesday morning joint session, President D. E. Smith on Tuesday afternoon and incoming President Miller on Wednesday afternoon at the sessions of the Association. The following papers were given. Abstracts of most of these follow, the numbers corresponding to the numbers in the lists of titles:

JOINT SESSION OF THE ASSOCIATION WITH SECTION L OF THE AMERICAN ASSOCIATION.

(1) "Geometrical development of analytical ideas" by Professor L. C. KARPINSKI, University of Michigan.

(2) "The anharmonic ratio in projective geometry" by Professor E. B. STOUFFER, University of Kansas.

(3) Introductory note on "The Association's ideal for expository papers" by Professor E. J. WILCZYNSKI, University of Chicago.

(4) "The first work on mathematics printed in the New World" by Professor D. E. SMITH, Columbia University.

(1) The purpose of this paper by Professor Karpinski is to show that many of the fundamental ideas of Greek geometry, and of all geometry up to the time of Newton, correspond closely to elementary analytical ideas. Even the proofs are frequently parallel to modern analytical proofs. The problem of the construction of the regular pentagon is closely connected with the historical development of a large portion of the first four books of Euclid; the problem reduces, of course, to the solution of a quadratic equation. The duplication of the cube, algebraically $x^3 = 2a^3$, was solved by the Greeks by two intersecting conics. Similarly the trisection of the angle, the regular seven- and nine-sided polygons were recognized by the Arabs as leading to cubic equations, and solved by conics. Even the squaring of the circle led the Greek Hippocrates to a problem on "application of areas," or quadratics. The problems of the infinitesimal and the theory of limits, with the Eudoxian "method of exhaustion" are also strictly analytical in statement.

(2) In line with the increased interest in the unifying concepts of mathematics, Professor Stouffer discussed the fundamental nature of the anharmonic ratio in projective geometry. The development of the anharmonic ratio concept from the time of Euclid was traced briefly and two theorems were stated which show that information concerning a projectivity is equivalent to information concerning anharmonic ratios. These theorems simplify the proof of many propositions of projective geometry. An illustration of this fact was given by the proof of an important theorem on conics. Several concepts of a fundamental nature in geometry were introduced by means of anharmonic ratios in order to show the value of the method.

(3) Professor Wilczynski's paper will appear in full in the April issue of the MONTHLY.

The Association was fortunate in having on the program four papers which were referred to by a number of members as excellent examples of the kind of expository papers which are possible and desirable.

(4) Professor Smith's paper was printed in the MONTHLY for January, 1921.

SESSION OF THE ASSOCIATION.

(5) "Rolle's theorem and its generalizations" by Professor A. J. KEMPNER, University of Illinois.

(6) "Some geometrical aspects of the theory of relativity" by Professor L. W. DOWLING, University of Wisconsin.

(7) Note on "The metric question from the historical standpoint" by Professor L. C. KARPINSKI, University of Michigan.

(5) Professor Kempner's paper dealt with extensions of the theorem that if a real polynomial equation, *i.e.*, an equation $f(x) = 0, f(x)$ a polynomial with real coefficients, has all of its roots real, then the roots of the derived equation are also all real and are separated by the roots of the given equation. He discussed the well-known Gauss-Lucas polygon theorem, according to which for any polynomial equation (with real or complex coefficients) the derived equation has all roots in the smallest convex polygon which can be described in the complex plane around the points representing the roots of the given equation. The extension of these theorems to other types of equations was considered, the various fields in mathematics in which Rolle's theorem is of importance were mentioned, and some outstanding problems were emphasized.

(6) The following is an outline of Professor Dowling's paper: I. *The Coördinate System.* (*a*) In primitive times. (*b*) In Ptolemy's day. (*c*) After Copernicus. (*d*) After Kepler and Newton. (*e*) After Lorentz;—the general quadratic form $g_{ik}\xi_i\xi_k$ and the associated bilinear form $g_{ik}\xi_i\eta_k$, together with the group of linear transformations which leave these forms invariant. A study of this group and its system of invariants and covariants constitutes the special theory of relativity.

II. *Minkowski Space-Time.* (*a*) The postulate of constant velocity of light; other known velocities less than or equal to that of light. (*b*) The formation of the "light-cone." (*x*) The quadratic form $\xi_1^2 + \xi_2^2 + \xi_3^2 - c^2\xi_4^2$ and the associated bilinear form $\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 - c^2\xi_4\eta_4$, together with the Lorentz transformation which leaves these forms invariant.

III. *Gravitation.* (*a*) The Principle of Equivalence. (*b*) A local gravitational field of force generated by the quadratic transformation $x = \bar{x}$, $y = \bar{y} + m\bar{x}^2$. (*c*) Quadratic Differential Forms $g_{ik}dx_idx_k$;—the necessary and sufficient conditions that two such forms shall be equivalent. (*d*) The Einstein Space-Time as compared with the Minkowski Space-Time;—their non-equivalence.

IV. The mathematical basis of the special theory of relativity lies in the work of Cayley, Sylvester, Salmon in England; Clebsch, Gordan, Aronhold in Germany; Hermite in France; Brioschi in Italy and many others.

The general theory of relativity rests upon the work of Gauss (1827), Lamé (1859), Riemann (1864), Christoffel (*Crelle* 1870), Ricci and Levi-Cività (*Math. Annalen* 1901), Lie (*Math. Ann.* 1884), Maschke (*Transactions A. M. S.* 1900–1903).

(7) This note by Professor Karpinski calls attention to the desirability that the Association actively enroll itself as supporting the meter-liter-gram system in the United States. A quotation from Simon Stevin, the first writer on decimal fractions, reveals the comprehensive appreciation on the part of Stevin of the great use of decimal fractions as applied to weights, measures, and money.

JOINT SESSION OF THE ASSOCIATION WITH THE AMERICAN MATHEMATICAL SOCIETY AND SECTIONS A AND L OF THE AMERICAN ASSOCIATION.

(8) "A decade of American mathematics," Retiring address as chairman of Section A, Professor O. D. KELLOGG, Harvard University.

(9) "Evolution of algebraic notations" by Professor FLORIAN CAJORI, University of California.

(8) The address of Professor Kellogg was a rapid survey of the distribution of the mathematical effort of the decade in America among the various branches of the subject. It contained pleas for greater development of mathematical physics through a more rational attitude of mathematician and physicist each toward the needs of the other; for the more general cultivation of a sense of values; for the development of a sense of obligation of the individual mathematician to support the publication of worthy American monographs, to produce to the best of his ability, and to give greater attention to making his own contributions and his science in general appeal as widely as possible. This address will be printed in *Science*.

(9) Professor Cajori exhibited slides made from early Italian, German, French and English books and manuscripts, as well as from seventeenth, eighteenth and nineteenth century text books, for the purpose of showing the struggle which has been going on between the rhetorical and the purely symbolic tendencies. From the experiences of the past, inferences were drawn which may serve as present and future guides to mathematicians on matters of algebraic symbolism.

SESSION OF THE ASSOCIATION.

(10) "General aspects of the problem of interpolation" by Professor DUNHAM JACKSON, University of Minnesota.

(11) "Construction of double entry tables" by Professor A. A. BENNETT, University of Texas, in charge of the U. S. Ordnance Ballistic Station, Baltimore, Md.

(12) "Certain general properties of functions" by Professor HENRY BLUMBERG, University of Illinois.

(10) In Professor Jackson's paper, it was pointed out that the problem of interpolation is primarily that of determining a function of specified form, most often a polynomial, which takes on given values for a certain number of given values of the independent variable; but that it is important to generalize the notion, at least to the extent of including functions determined by a finite number of given values, whether coinciding absolutely with those values or not. There are indicated then some of the striking analogies, both formal and more profound, between the formulas of interpolation by polynomials and finite trigonometric sums on the one hand, and Taylor's and Fourier's series on the other.

(11) The double entry tables considered by Professor Bennett were tables with numerical entries, which may be regarded as the values of a function $F(x, y)$ for equally spaced intervals of x and equally spaced intervals of y. The original

data are supposed to be (1) not entirely reliable, (2) possible to obtain by observation or computation for preassigned values of the independent variables, but only with difficulty, (3) not subject to any known formal law. The implications of these conditions were discussed. The advantages of using throughout fourth order differences were pointed out in some detail. Finally the use of "central" differences was explained in the construction of double entry tables.

(12) The substance of Professor Blumberg's paper was largely drawn from his 1917 *Annals* paper and his communication to the American Mathematical Society, April, 1919. No technical knowledge beyond the calculus was presupposed from the hearers and the presentation was couched in expository form.

Meeting of the Board of Trustees of the Association.

Eight members were present at each session.

The following 73 persons and 3 institutions, on applications duly certified, were elected to membership:

To individual membership.

Anna H. Andrews, Ph.B. (Wesleyan). Teacher, Public High School, Hartford, Conn.

B. I. Baidaff, Licentiat in mat. (Univ. of Jassy, Roumania). Buenos Aires, Argentina.

J. P. Ballantine, A.B. (Harvard). Instr., Penn. State Coll., State College, Pa.

P. E. Basye, Univ. of Missouri, Columbia, Mo.

E. M. Berry, Ph.D. (Iowa). Instr., Purdue Univ., W. LaFayette, Ind.

Emile Borel, D.Sc. (Paris). Professeur à la Faculté des Sciences de Paris.

J. W. Calhoun, A.M. (Harvard). Asso. prof. of appl. math., Univ. of Texas, Austin, Tex.

R. H. Carpenter, A.M. (Kansas). Instr., Univ. of Kansas, Lawrence, Kans.

C. M. Cleveland, B.E. in Civil Engg. (Mississippi). Instr. in appl. math., Univ. of Texas, Austin, Tex.

R. P. Conkling, A.B. (Cornell). Head of dept. of math., Tech. School; asst. in math., Central C. and M. T. High School, Newark, N. J.

A. E. Cooper, E. E. (Texas). Instr. in appl. math., Univ. of Texas, Austin, Tex.

Odyne O. Cornell, A.B. (Nebraska). Mangum, Okla.

M. E. Cox, B.S. in M.E. (Clemson Coll.). Asst. prof., Texas A. and M. Coll., College Station, Tex.

A. S. Croom, A.B. (Louisville). Acting dean, Harper Coll., Harper, Kans.

Julia Dale. A.B. (Transylvania Coll.). Instr., Univ. of Missouri, Columbia, Mo.

Alice C. Dean, A.M. (Rice Inst.). Fellow and acting libr., Rice Inst., Houston, Tex.

Mary E. Decherd, A.M. (Texas). Instr., Univ. of Texas, Austin, Tex.

S. Dickstein. Prof., Univ. of Warsaw, Warsaw, Poland.

L. H. DUBE, Ph.D., D.D. (Gregorian Univ., Rome); M.Sc. (Ottawa). Prof. of higher math., Ottawa Univ., Ottawa, Can.

J. R. EVERETT, A.M. (Wisconsin). Instr., Carnegie Inst. of Tech., Pittsburgh, Pa.

LUCY A. FEDDERSEN, A.B. in Educ. (Wyoming). Instr., Univ. of Wyoming High School, Laramie, Wyo.

A. R. FEHN, Ph.B. (Baldwin-Wallace Coll.). Asso. prof., Univ. of Wyoming, Laramie, Wyo.

FLORENCE E. FIELD, A.M. (Michigan). Acting head of dept., Park Coll., Parkville, Mo.

R. M. FOSTER, B.S. (Harvard). Dept. of development and research, Amer. Tel. and Tel. Co., New York, N. Y.

J. G. FOWLKES, A. B., B.O. (Ouachita Coll.). Head of dept. of math., Roger Ascham School, New York, N. Y.

H. J. GAY, A.B. (Harvard). Instr., Worcester Polytech. Inst., Worcester, Mass.

J. S. GOLD, B.S. (Bucknell). Instr., Bucknell Univ., Lewisburg, Pa.

P. H. GRAHAM, A.M. (Virginia). Instr., New York Univ., New York, N. Y.

H. H. HAMMER, A.M. (Texas). Asst. to state actuary, Austin, Tex.

A. J. HARGETT, A.M. (Transylvania Coll.). Head of dept. of math., Texas Christian Univ., Fort Worth, Tex.

A. S. HATHAWAY, B.S. (Cornell). Prof., (retired), Rose Polytech. Inst., Terre Haute, Ind. Houston, Tex.

T. B. HENRY, A.B. (Kansas). Instr., Univ. of Kansas, Lawrence, Kans.

MABEL M. HEREN, M.S. (Northwestern Univ.). Asst. prof., Knox Coll., Galesburg, Ill.

T. F. HOLGATE, Ph.D. (Clark), LL.D. (Illinois; Queens). Prof., Northwestern Univ., Evanston, Ill.

HELMA L. HOLMES, A.M. (Nebraska). Instr. in pure math., Univ. of Texas, Austin, Tex.

B. P. HOOVER, A. B. (Baker Univ.), Asst. prof., Baker Univ., Baldwin, Kans.

GOLDIE P. HORTON, Ph.D. (Texas). Instr. in pure math., Univ. of Texas, Austin, Tex.

C. G. JAEGER, A.B. (Missouri). Instr., Univ. of Missouri, Columbia, Mo.

L. M. KLAUBER, A.B. in E.E. (Stanford). Genl. supt., Cons. Gas and Electric Co., San Diego, Calif.

J. J. KNOX, A.B. (Chicago). Instr., School of Eng. of Milwaukee, Milwaukee, Wis.

F. A. LA MOTTE, M.S. (Chicago); A.M. (Wisconsin). Instr., Junior Coll., St. Joseph, Mo.

G. L. LOWRY, B.S. (Bucknell). Instr., Bucknell Univ., Lewisburg, Pa.

ISRAEL MAIZLISH, M.S. (Mass. Inst. of Tech.). Instr. in physics, Reed Coll., Portland, Ore.

C. E. MELVILLE, A.B. (Northwestern). Asso. prof. and registrar, Coll. dept., Clark Univ., Worcester, Mass.

D. H. MENZEL, A.B. (Denver). Instr., Univ. of Denver, Denver, Colo.

A. D. MICHAL, A.B. (Clark Coll.). Fellow in math. and asst. in physics, Clark Univ., Worcester, Mass.

J. N. MICHIE, A.M. (Michigan). Adj. prof. of appl. math., Univ. of Texas, Austin, Tex.

PEARL C. MILLER, A.M. (Stanford). Asst., Washington Univ., St. Louis, Mo.

FANNIE S. MITCHELL, A.B. (N. Car. Coll. for Women). Teacher, High school, Gastonia, N. C.

ANNA M. MULLIKIN, A.M. (Pennsylvania). Instr., Univ. of Texas, Austin, Tex.

W. L. PHINNEY, B.S. (Dartmouth). Instr., Worcester Polytech. Inst., Worcester, Mass.

A. D. PIERSON, B.S. in Ed. (Missouri). Teacher, N. E. High School, Kansas City, Mo.

C. S. PORTER, A.B. (Amherst). Instr., Worcester Polytech. Inst., Worcester, Mass.

HUGH PORTER, A.M. (Texas). Asso. prof., N. Texas St. Normal Coll., Denton, Tex.

CAROLINE M. REAVES, A.M. (Oklahoma). Prof., Coker Coll., Hartsville, S. C.

N. B. ROSENBERGER, A.M. (Pennsylvania). Grad. student, Teachers Coll., Columbia Univ., New York, N. Y.

JEAN F. ROSS. Librarian, High school, Sacramento, Calif.

BERNICE SANDERS, A.B. (Wilberforce). Prof., Wilberforce Univ., Wilberforce, Ohio.

E. W. SCHREIBER, A.B. (Michigan). Head of dept. of math., Proviso Township High School, Maywood, Ill.

PINCAS SCHUB, formerly student in Turkey. Fellow, Clark Univ., Worcester, Mass.

JABIR SHIBLI, A.M. (North Dakota). Prof., Fargo Coll., Fargo, N. Dak.

F. L. SMITH, A.B. (Drury Coll.) Asst. in physics, Univ. of Missouri, Columbia, Mo.

R. F. SMITH, M.S. (New York Univ.). Asst. prof., Coll. of City of New York, New York, N. Y.

G. W. SNEDECOR, A.M. (Michigan). Asso. prof., Iowa State Coll., Ames, Ia.

MAY J. SPERRY, A.M. (Brown). Instr., Knox Coll., Galesburg, Ill.

LOUISE E. C. STUERM, A.M. (Columbia). Dept. of development and research, Amer. Tel. and Tel. Co., New York, N. Y.

J. A. SWENSON, A.B. (Columbia). Head of dept. of math., Wadleigh High School, New York, N. Y.

O. B. TROUT, A.M. (Denver). Prof. of math. and registrar, Univ. of Denver, Denver, Colo.

E. R. TUCKER, B.S. (Milit. Coll. of S. C.); A.B. (Texas Chr. Univ.). Asso. prof., Texas Chr. Univ., Fortworth, Texas.

F. N. WELLS, Instr., U. S. Naval Acad., Annapolis, Md.

Mrs. F. E. WOOD, A.B. (Baker Univ.). Fellow, Univ. of Kansas, Lawrence, Kans.

P. W. WOOD, M.A. (Cambridge). Fellow, tutor, librarian and mathematical lecturer, Emmanuel Coll., Cambridge, Eng.
FRANCES W. WRIGHT, A.M. (Brown). Instr., Elmira Coll., Elmira, N. Y.

To institutional membership.

SAINT BENEDICT'S COLLEGE, Atchison, Kansas.
UNIVERSITY OF CINCINNATI, Cincinnati, Ohio.
EAST TEXAS STATE NORMAL COLLEGE, Commerce, Tex.

The Board granted authority to the Association members residing in Texas to organize a Texas Section. Professor H. J. ETTLINGER has been elected chairman and Professor J. L. RILEY secretary-treasurer of the temporary organization.

The nomination of Professor E. R. HEDRICK, which had been made for one year from June 11, 1920, as the Association's representative on the Executive Committee of the Division of Physical Sciences, National Research Council, was extended to a term of three years from that date.

The Board voted to present the name of Professor E. H. MOORE to the Council of the American Association for nomination to the office of president for 1921. Supported by similar action taken by other affiliated organizations, Professor Moore was elected to the presidency at the Thursday morning meeting of the Council.

At the meeting of the incoming Board of Trustees Wednesday afternoon, the following were appointed Associate Editors of the MONTHLY for the year 1921:

ALBERT A. BENNETT,	WALTER B. FORD,	ULYSSES G. MITCHELL,
EDWARD L. DODD,	CUTHBERT F. GUMMER,	CHARLES N. MOORE,
OTTO DUNKEL,	HENRY P. MANNING,	DAVID E. SMITH,
BENJAMIN F. FINKEL,	RAYMOND B. MCCLENON,	HORACE S. UHLER.

Other business was outlined in reference to the office of the editor-in-chief and with regard to the basis for exchange in the payment of dues by foreign members.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION.

The secretary-treasurer announced the names of those elected to membership by the Board. He reported also the death, in 1920 with one exception, of the following members:

MAELYNETTE ALDRICH, Professor of mathematics, Martha Washington College (February 22);
G. E. FISHER, Professor of mathematics, University of Pennsylvania (March 28);
Y. H. HO, Fellow, graduate school, Cornell University (February 22);
R. S. LAWRENCE, Professor of mathematics, Hanover College (January 30, 1919);
O. A. RANDOLPH, Associate professor of physics, University of Colorado (April 11);
WILLIAM RINCK, Professor of mathematics, Theological School and Calvin College (November 11);
E. W. STANTON, Dean, Iowa State College (September 12).

On motion of Professor Karpinski the following resolution was voted:

"Resolved, that the Mathematical Association of America favors the national use of the meter-liter-gram system. The Association respectfully urges upon Congress that steps be taken to make the use of this system national in character and in particular that scientific departments of the government be required to use this system.

The election of officers for the year 1921 was conducted by mail and in person at this meeting, as provided in the By-Laws. The tellers (C. N. MOORE and W. L. HART) reported the result of the balloting, the noteworthy number of 454 ballots having been cast:

For President: G. A. Miller, 246 votes; E. J. Wilczynski, 208 votes.

For Vice-Presidents: R. C. Archibald, 307 votes; R. D. Carmichael, 247 votes; Elizabeth B. Cowley, 120 votes; Helen A. Merrill, 224 votes.

For additional members of the Board of Trustees (to serve until January, 1924): A. A. Bennett, 205 votes; W. H. Bussey, 163 votes; Florian Cajori, 358 votes; E. L. Dodd, 107 votes; C. F. Gummer, 175 votes; H. L. Rietz, 283 votes; W. H. Roever, 176 votes; D. E. Smith, 318 votes.

The following were accordingly declared elected:

President: G. A. MILLER, University of Illinois.

Vice-Presidents: R. C. ARCHIBALD, Brown University; R. D. CARMICHAEL, University of Illinois.

Additional members of the Board of Trustees: A. A. BENNETT, University of Texas; FLORIAN CAJORI, University of California; H. L. RIETZ, University of Iowa; D. E. SMITH, Columbia University.

Because of the election of Professor Carmichael as vice-president, the Board of Trustees, in exercise of its constitutional authority, appointed to the vacancy in the Board for the term ending January, 1923, Professor C. F. GUMMER of Queen's University, as a representative of our Canadian constituency.

The secretary-treasurer made his financial report for the year, giving an account of all business transacted for the Association up to December 2, 1920. The report of the auditing committee (Mrs. MAYME I. LOGSDON, H. E. SLAUGHT, and F. M. MCGAW, chairman) was then made. The financial report is printed in full below.

If to the balance on 1920 business shown in this report, $3,234.37, there be added the amount of bills receivable, $165.00, and there be subtracted the estimated amount of bills payable, $2,040.00, there results an estimated final balance on 1920 business of approximately $1,360. It will be recalled that, of this surplus, about $1,000 was passed over to the association by the management of the MONTHLY when the Association was organized five years ago, and this fund is kept by the Board of Trustees of the Association as a reserve fund. If the estimated balance of $1,360 be compared with the corresponding figure of one

year ago, $2,040 (See MONTHLY, March, 1920, p. 112), it will be seen that there is a probable deficit on the year's business of nearly $700, a fact forecast in the report made at the summer meeting and noted in the MONTHLY for November.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DECEMBER 2, 1920.

RECEIPTS.			EXPENDITURES.		
Balance Dec. 15, 1919		$4,581.07	Publisher's bills (Sept. '19–Oct. '20)		$4,720.10
1917–19 indiv. dues	$ 147.90		President's office		46.16
1919 instit. dues	7.25		Manager's office		22.19
1919 subscriptions	3.00		Editor-in-chief's office		585.45
1920 indiv. dues	3,171.66		Other editors' postage		10.10
1920 instit. dues	359.30		Secretary-Treasurer's office:		
1920 subscriptions	517.85		Postage	$ 99.77	
Initiation fees	324.00		Bond	5.00	
Sale copies of MONTHLY	104.92		Office supplies	24.48	
Sale reprints	1.95		Express, telegrams, etc.	10.27	
Advertising	632.52		Clerical work	217.75	
Exchange	.50		Printing 1919 Register	269.62	
Interest State Savgs. Bk.	78.20		Printing	215.76	
Interest Peoples Bk.	58.13		New York meeting	106.16	
Interest Liberty Bonds	32.49		Chicago summer meeting	43.00	
			Paid to sections from initiation fees	59.31	
					$1,051.12
Total 1920 receipts		$5,439.67	*Annals* subvention		275.00
			Services and fees for incorporation		76.25
Total assets up to 1921 business		$10,020.74	Total expenditures		$6,786.37
Total expenditures		6,786.37			
			Cash on hand		$ 20.64
Balance to the end of 1920 business		$3,234.37	Checking account		601.50
			State Savgs. Bk. Co. account		1,247.48
			Peoples Bkg. Co. account		912.14
Received on 1921 business (including $33 contributed to 1921 expenses)		547.39	Liberty Bond		500.00
			Victory Bond		500.00
Book balance Dec. 2, 1920		$3,781.76	Bank balance Dec. 2, 1920		$3,781.76

When the accounts were closed on December 2, 1920, in order to furnish the auditing committee a complete record, there remained on the total business for the year 1920 the following items:

BILLS RECEIVABLE.		BILLS PAYABLE. (Either paid in December or estimated.)	
Advertising	$ 80.00	Publisher's bills (Nov.–Dec.)	$1,250.00
1920 dues unpaid	75.00	December *Annals* subvention	50.00
Interest Liberty Bonds	10.00	Init. fees due to sections	100.00
	$165.00	President's office	50.00
		Manager's office	20.00
		Editor-in-chief's office	150.00
		Secretary-treasurer's office	125.00
		Printing prelim. and annual ballots, program, etc.	225.00
		Additional postage	70.00
			$2,040.00

On account of the financial outlook a special letter was sent to the members of the Association in October, announcing the new membership dues and the need for subsidiary funds. It is our hope that the appeal for special contributions for 1921 expenses will not be forgotten as members pay their dues, for it is chiefly on this that we rely to offset the decrease in the reserve fund of the Association. It is gratifying to note here that a generous gift of $45.50 from President Smith and of $10.00 from Teachers College covers the expense of an effective campaign for members in America and in Europe, an expense that would otherwise fall directly on the Association treasury. It should also be noted that of 400 members who had paid their dues for 1921 before January first, sixty have made special contributions ranging from one up to seventeen dollars, a total thus far of $150.

While doubtless some may feel that they cannot do more than to pay the increased dues, it is within the range of possibility for almost any member to secure at least one new member; this in itself is a very helpful contribution to the finances of the Association and a real service to those not yet affiliated with the Association.

W. D. CAIRNS, *Secretary-Treasurer.*

ACOUSTIC CIRCLES.

By H. M. DADOURIAN, Trinity College.

The determination of the position of enemy artillery by *sound ranging* is one of the most interesting examples of the application of science to modern warfare. This consists in observing the time of arrival of the report, or the muzzle wave, of the enemy gun at a number of observation posts of known positions and then in using the *phonotelemetric* data thus obtained to determine the position of the gun. One of the simplest methods used in sound ranging is the following, known as the *method of concentric circles.*

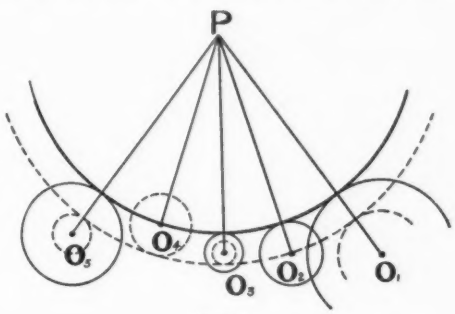

FIG. 1.

Let P, Fig. 1, denote the position of the enemy gun; O_1, O_2, etc., the positions of the observation posts at which the apparatus used for registering the arrival of the sound wave is placed; and T_1, T_2, etc., the times at which the sound wave reaches the posts O_1, O_2, etc., respectively. Furthermore suppose P, O_1, O_2, etc. to be in the same plane, with no intervening objects to obstruct the free propagation of the wave with uniform and constant velocity. Then it is evident that when a spherical wave originating at P reaches the nearest post O_i (O_4 in

the case represented by Fig. 1) its distances from O_1, O_2, etc. will be $V(T_1 - T_i)$, $V(T_2 - T_i)$, etc., respectively, where V denotes the velocity of propagation of the wave. Therefore the circle about P passing through O_i is tangent to circles drawn about O_1, O_2, etc., having radii

$$\begin{aligned} R_{1i} &= V(T_1 - T_i), \\ R_{2i} &= V(T_2 - T_i), \\ &\cdots\cdots, \end{aligned} \tag{1}$$

respectively. This leads immediately to the method of concentric circles. On a drawing board points are located representing accurately the relative positions of the observation posts on a given scale, say 1 in 20,000; also a transparent sheet with closely drawn concentric circles is provided. When T_1, T_2, etc., are obtained circles are drawn about O_1, O_2, etc., with radii given by equations (1), the sheet with concentric circles is passed over the table until one of its circles becomes equally near tangency to all the circles about the points O_1, O_2, etc. Then the center of the concentric circles represents the position of the origin of the sound wave.

If in equations (1) O_i is not the post nearest P, then the right hand members of some of these equations become negative and the circle about P becomes tangent externally or internally to the circle about any post O_j according as $(T_j - T_i)$ is positive or negative, respectively. If in Fig. 1 O_2 is taken as the reference post, the circle passing through O_2 and having its center at P is tangent internally to the circles about O_3 and O_4, while it is tangent externally to the circles about O_1 and O_5 as is shown by the broken curves. Evidently the circle about the reference point has zero radius, consequently the tangency may be considered either as internal or as external or both.

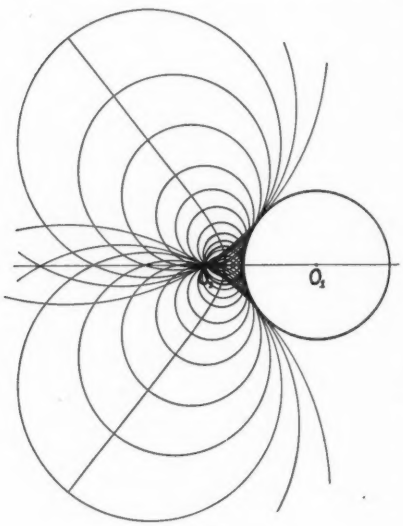

FIG. 2.

It is the object of this paper to show that the circle about P is common to a number of systems of circles which are determined by a set of phonotelemetric data. Consider two observation posts O_1 and O_2, Fig. 2, at which the arrival of a spherical sound wave is registered at the instants T_1 and T_2, respectively. Let r_1 and r_2 be, respectively, the distances of O_1 and O_2 from the origin of the wave; and t_1 and t_2 the intervals of time taken by the wave to reach the points O_1 and O_2, then

$$\begin{aligned} r_1 &= Vt_1, \\ r_2 &= Vt_2, \end{aligned}$$

and

$$r_1 - r_2 = V(t_1 - t_2).$$

But

$$t_1 - t_2 = T_1 - T_2.$$

Therefore

$$r_1 - r_2 = V(T_1 - T_2). \qquad (2)$$

Since the right-hand member is constant, equation (2) is the well-known equation of a hyperbola of which

$$R_{12} = V(T_1 - T_2)$$

is the transverse axis, and O_1 and O_2 are the foci. Thus the origin of the wave may be anywhere on the hyperbola of which (2) is the equation. Therefore representing, by means of a circle, the wave which starts from any point of the hyperbola and which reaches the posts O_1 and O_2 at T_1 and T_2, respectively, we obtain a system of circles which pass through O_2, are tangent to the circle of radius R_{12} drawn about O_1 as center, and have their centers on the hyperbola of which (2) is the equation. Equation (2) may also be regarded as the *acoustic equation* of the system of circles, and the system of circles may be called *acoustic circles*. The origin of the wave then is at the center of one of the circles of the system of acoustic circles of which (2) is the acoustic equation. If $T_1 > T_2$, the circles of

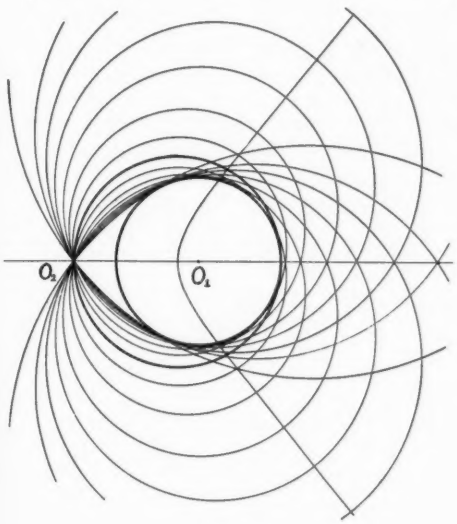

FIG. 3.

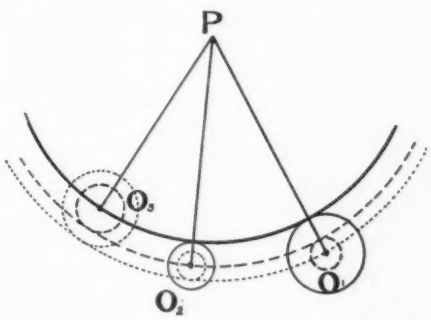

FIG. 4.

the system have their centers on the left-hand branch of the hyperbola as in Fig. 2 and are tangent to the circle around O_1 externally. If on the other hand $T_1 < T_2$, the circles have their centers on the right-hand branch and are tangent to the circle around O_1 internally, as in Fig. 3. If the position of the source of sound is not limited to a plane, the system of acoustic circles becomes a system of spheres which have their centers on the hyperboloid of revolution obtained by revolving the hyperbola about the line O_1O_2 and which are tangent to the sphere of radius R_{12} drawn about O_1 as center.

Now let T_3 be the time at which the wave front reaches another observation post O_3, then considering the data of two of the three posts at a time we obtain the equations

$$\begin{aligned} r_1 - r_2 &= V(T_1 - T_2), \\ r_1 - r_3 &= V(T_1 - T_3), \\ r_2 - r_3 &= V(T_2 - T_3), \end{aligned} \tag{3}$$

and

which are the acoustic equations of three systems of circles and of as many hyperbolas. The origin of the wave is at the common center of three circles, Fig. 4, one from each system. Each of the three circles passes through one of the points O, is tangent to circles about the other two and has its center at the common point of intersection of the hyperbolas given by equations (3).

In general, if the times of arrival of a wave at n posts of known positions are given, $n(n-1)/2$ equations can be obtained from the data, each of which forms the equation of a system of acoustic circles. (The truth of this statement may be easily seen by considering the number of permutations and combinations of n things taken two at a time). Evidently there are two circles in each system whose centers are at the origin of the wave. Of the $n(n-1)$ circles having their centers at the origin of the wave $(n-1)$ pass through O_1 and consequently are coincident; $(n-1)$ pass through O_2 and are coincident; and so on to the $(n-1)$ which pass through O_n. Therefore in general there are n distinct circles in the $n(n-1)/2$ systems which have their centers at the origin of the wave. The circle which passes through any post O_i is tangent to a circle of radius $V(T_j - T_i)$ drawn about the post O_j as center. The tangency is external if $T_j > T_i$ and internal if $T_j < T_i$.

ON THE NUMERICAL VALUE OF i^i.

By H. S. UHLER, Yale University.

The primary object of the present note is to place on record the values of $e^{-\pi/2}$ and of seven related powers of the same base which I have recently calculated to more than fifty decimal places.

In order to avoid the use of any table of mathematical constants the following series was employed

$$y \equiv e^{\sin^{-1} x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \cdots [x^2 < 1]$$

$$= (1 + \sum_1^\infty t_{2k+1}) + (x + \sum_1^\infty t_{2k+2}).$$

Since the collections of series at my disposal did not contain a formula for the general term of the above series it was necessary to prove that

$$t_{2k+1} = (0^2 + 1)(2^2 + 1)(4^2 + 1) \cdots [(2k-2)^2 + 1]x^{2k}/(2k)!$$

and

$$t_{2k+2} = (1^2 + 1)(3^2 + 1)(5^2 + 1) \cdots [(2k - 1)^2 + 1]x^{2k+1}/(2k + 1)!,$$

where $k = 1, 2, 3, \cdots$. This was easily accomplished by the aid of Leibnitz's theorem in the manner illustrated in Williamson's *Differential Calculus*, 1889, pages 51–56; the fundamental equation being

$$(1 - x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n + 1)x\frac{d^{n+1}y}{dx^{n+1}} - (n^2 + 1)\frac{d^n y}{dx^n} = 0.$$

For $x = \pm 1/2$, $y = e^{\pm\pi/6}$.

With $x = 1/2$ the terms involving even powers of x were calculated independently of the odd-power terms. Since, in both sets, each term was derived directly from the preceding one, the arithmetical operations involved had to be checked very carefully by the inverse operations, for a single incorrect digit would have vitiated some or all of the succeeding terms in the set in which the error was committed. The total number of even terms used was the same as that of the odd terms, namely 85. Each term was computed to the nearest unit in the fifty-fourth decimal place, 5×10^{-55} being counted as 1×10^{-54}. $e^{\pm(\pi/3)}$ was obtained by squaring $e^{\pm(\pi/6)}$, $e^{\pm(\pi/2)}$ by multiplying together $e^{\pm(\pi/3)}$ and $e^{\pm(\pi/6)}$, and $e^{\pm\pi}$ by squaring $e^{\pm(\pi/2)}$. Let σ_1 and σ_2 denote respectively the numerical values actually found for the sum of the even terms and of the odd terms respectively when $x = 1/2$. The results of the calculation were:

$$\sigma_1 = 1.14023\,83210\,76428\,79214\,11319\,80379\,35089\,07668\,97667\,51132\,0551,$$

$$\sigma_2 = 0.54785\,34738\,88039\,80847\,57156\,47717\,43140\,33527\,83521\,77586\,1757,$$

$$e^{\frac{\pi}{6}} = 1.68809\,17949\,64468\,60061\,68476\,28096\,78229\,41196\,81189\,28718\,2308,$$

$$e^{-\frac{\pi}{6}} = 0.59238\,48471\,88388\,98366\,54163\,32661\,91948\,74141\,14145\,73545\,8794,$$

$$e^{\frac{\pi}{3}} = 2.84965\,39082\,26361\,49747\,41273\,19852\,90439\,39640\,06102\,78112\,6866,$$

$$e^{-\frac{\pi}{3}} = 0.35091\,98071\,78410\,96756\,57367\,15996\,95305\,83625\,73153\,62096\,1747,$$

$$e^{\frac{\pi}{2}} = 4.81047\,73809\,65351\,65547\,30356\,66703\,83312\,63901\,70874\,66453\,4901,$$

$$e^{-\frac{\pi}{2}} = i^i = 0.20787\,95763\,50761\,90854\,69556\,19834\,97877\,00338\,77841\,63176\,9614,$$

$$e^{\pi} = 23.14069\,26327\,79269\,00572\,90863\,67948\,54738\,02661\,06242\,60021\,16,$$

$$e^{-\pi} = 0.04321\,39182\,63772\,24977\,44177\,37171\,72801\,12757\,28109\,81063\,30854.$$

The reliability of the preceding numbers was tested by checks applied independently to the final results. Thus $s_1^2 - s_2^2 = 1$, where s_1 and s_2 are the exact values approximated above by the corresponding expressions in Greek letters, was used to check the values of those expressions. It is concluded that the calculated values of $e^{\pi/6}$ and $e^{-(\pi/6)}$ given above are certainly correct to 52 decimal places and they are probably in error by not more than two units in the fifty-

third place. It was also found that to fifty-two places of decimals neither $e^{\pi} \cdot e^{-\pi}$ nor $e^{\pi/2} \cdot e^{-(\pi/2)}$ differed from unity.

In this MONTHLY, *1917*, 237, it is stated that—for the value of $e^{-(\pi/2)}$ or i^i—"Mr. Escott using Steinhauser's 20-place tables, gets .20787957635076190854687 while Professor Reynolds, using Hutton's 20-place tables, gets .2078795763-4917907781." Comparison of these numbers with my datum given above shows that Mr. Escott's differs from it by only about one unit in the twenty-second decimal place, whereas Professor Reynolds's number is discordant at, and beyond, the eleventh place. Since this may mean that there is an error in Hutton's tables it would be helpful if Professor Reynolds would investigate and report the cause of the discrepancy.

Finally, in the seventh revised edition of the *Smithsonian Physical Tables*, 1920, page 55, a short table of values[1] of $e^{(\pi/4)x}$, $e^{-(\pi/4)x}$ and their logarithms is given. If it should ever become necessary to compute tables of this kind to a greater number of decimal places than can be effected advantageously by logarithms, my data given above may be used either as basic numbers or as independent checks.

HISTORICAL NOTES ON THE RELATION $e^{-(\pi/2)} = i^i$.

BY R. C. ARCHIBALD, Brown University.

In 1719 Count Giulio Carlo de'Toschi di Fagnano showed, in effect, that the arc of the quadrant of a unit circle $(\pi/2)$ is[2]

$$2 \log.(1 - \sqrt{-1})^{\frac{1}{2}\sqrt{-1}} \times (1 + \sqrt{-1})^{-\frac{1}{2}\sqrt{-1}}.$$

---

[1] For $x = 1, 2, 3, \cdots 20$. The part of the table without logarithms is also given on page 91 of J. B. Dale, *Five Figure Tables*. London, 1903. The values of $e^{19\pi/4}$ differ materially in these two sources.—EDITOR.

[2] *Opere Matematiche del Marchese . . . de'Toschi di Fagnano.* Milano, volume 2, 1912. On page 406 we find "$\int \frac{dt}{1+t^2}$ esprime l'arco di cerchio [radius unity], la di cui tangente é t" [if the equations of the circle are $x = \cos\theta$ and $y = \sin\theta$, $t = \tan\theta$]. On pages 422–423 we find the following:

$$\int \frac{dt}{1+t^2} = \log.\ (1 - t\sqrt{-1})^{\frac{1}{2}\sqrt{-1}} \times (1 + t\sqrt{-1})^{-\frac{1}{2}\sqrt{-1}}$$

or

"(8) $$\int \frac{dt}{1+t^2} = \log.\ (A^2 - B^2).$$

[where $A = 1 + \frac{1}{2}t + \frac{1}{8}t^2 - \frac{7}{48}t^3 - \frac{43}{384}t^4 \cdots$, $\quad B = \frac{1}{4}t^2\sqrt{-1} + \frac{1}{8}t^3\sqrt{-1} - \frac{3}{32}t^4\sqrt{-1} \cdots$].

"Queste due ultime equazioni manifestano una nuova, e bellissima proprietà del cerchio, ciascun arco del di cui quadrante à per suo elemento $\frac{dt}{1+t^2}$, quando la t denota la tangente dell'arco medesimo.

"Scolio IV.—Se l'arco di cerchio fosse equale al quadrante, allora la t diverrebbe infinita, e per avere il logaritmo equale al quadrante nulla gioverebbe l'equazione (8). In questo caso si divida per mezzo lo stesso quadrante, e la tangente dell'arco sudduplo di esso sarà eguale all'unità;

The relation $\pi/2 = -\sqrt{-1}\log\sqrt{-1}$ was known to Euler as early as 1728, since, on December 10 of that year he wrote as follows to Jean Bernoulli[1]: "Sit radius circuli a, sinus y, cosinus x, erit ex methodo tuâ quadraturam circuli ad logarithmos reducendi, area sectoris $= \frac{aa}{4\sqrt{-1}}\log\frac{x+y\sqrt{-1}}{x-y\sqrt{-1}}$, et posito $x = 0$, habebis quadrans circuli $\frac{aa}{4\sqrt{-1}}\log(-1)$." Hence, Euler demonstrated that $\frac{\pi a^2}{4} = \frac{a^2}{4\sqrt{-1}}\log(-1)$, from which one readily deduces

$$\frac{\pi}{2} = -\sqrt{-1}\log\sqrt{-1}.$$

In his reply, Jean Bernoulli called attention[1] to the identity

$$\int_0^{\pi/4}\frac{a^2dx}{2\sqrt{a^2-x^2}} = \frac{a^2}{4}\frac{\log\sqrt{-1}}{\sqrt{-1}},$$

and observed at the same time that the integral is equal to one eighth of a circle of radius a. It follows immediately that

$$\frac{\pi a^2}{8} = \frac{a^2}{4}\frac{\log\sqrt{-1}}{\sqrt{-1}}, \text{ or } \frac{\pi}{2} = -\sqrt{-1}\log\sqrt{-1};$$

but Jean Bernoulli did not himself draw this conclusion because he believed that $\log\sqrt{-1}$ was zero.[2]

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pongasi poscia 1 in luogo di t nel secondo membro dell'equazione (8), e si avrà il logaritmo eguale all'arco sudduplo del quadrante, e sarà log. (A^2-B^2). Quindi si avrà lo stesso quadrante $-2\,l.(A^2-B^2) = \log.(A^2-B^2)^2$."

In some notes written by Count Fagnano's son, G. F. Fagnano (1715–1797) in 1761, the following formula occurs (*Opere*, tome 3, 1912, p. 30):

$$\text{Quadrante} = 2\,L.\left(\frac{1-1\sqrt{-1}}{1+1\sqrt{-1}}\right)^{\frac{1\sqrt{-1}}{2}}.$$

It is shown that this reduces to

$$\text{"Quadrante} = 2\,L.(-1\sqrt{-1})^{\frac{1\sqrt{-1}}{2}} = \sqrt{-1}\,L.-\sqrt{-1}$$

la qual'Espressione con cede in bellezza alla Bernulliana." On page 34 is the following corollary:

"E facile dedurre da tale dottrina, che quantunque L. -1 abbia un'infinità di valori tutti immaginarj; siccome nell'Equazione v.g.

$$\frac{\text{Circonf.}}{\text{Diam.}} = \frac{\text{L.}-1}{\sqrt{-1}}$$

uno solo è il valore della Circonferenza, uno solo è il valore del Diametro, e uno solo il valore della radice di -1, così uno solo degli infiniti valori di L -1 salva la suddetta equazione."

[1] G. Eneström, *Bibliotheca Mathematica*, 1899, p. 46.

[2] This is discussed at length by Euler in "De la controverse entre Mrs. Leibnitz & Bernoulli sur les Logarithmes des nombres négatifs et imaginaires" (presented to the Berlin Academy in 1747), *Mém. de l'acad. d. sc. de Berlin*, vol. 5 (1749), 1751, pp. 146–148. On page 147 we find: "Or M. Bernoulli ayant si hereusement réduit la quadrature du cercle aux logarithmes des nombres

The discovery by Euler in this connection that $\log n$ has an infinite number of logarithms, which are all imaginary except when n is a positive number, is a striking indication of his clear thinking and genius.[1] In the course of his paper "Recherches sur les racines imaginaires des équations" [2] we find (pages 272–276) a discussion of the problem: "Une quantité imaginaire étant élevée à une puissance dont l'exposant est aussi imaginaire, trouver la valeur imaginaire de cette puissance." Assuming

$$(a + b\sqrt{-1})^{m+n\sqrt{-1}} = x + y\sqrt{-i}$$

he finds

$$x = c^m e^{-2\lambda n\pi - n\phi} \cos(2\lambda m\pi + m\phi + n \log c),$$

$$y = c^m e^{-2\lambda n\pi - n\phi} \sin(2\lambda m\pi + m\phi + n \log c),$$

where λ is a positive or negative integer, $c = \sqrt{(a^2 + b^2)}$ and $\cos(2\lambda\pi + \phi) = a/c$, $\sin(2\lambda\pi + \phi) = b/c$.[3] Euler's fourth corollary to this result is as follows:

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imaginaires, si le logarithme de $\sqrt{-1}$ etoit $= 0$, toute cette belle découverte seroit fausse; par laquelle il a fait voir, que le rayon est à la quatrieme partie de la circonference, comme $\sqrt{-1}$ à $\log\sqrt{-1}$. Donc posant le rapport du diametre à la circonference $= 1 : \pi$, il sera $\frac{1}{2}\pi = \dfrac{\log\sqrt{-1}}{\sqrt{-1}}$, & pourtant $\log\sqrt{-1} = \frac{1}{2}\pi\sqrt{-1}$, ce qui seroit absurde s'il étoit $\log\sqrt{-1} = 0$. Il n'est pas donc vray que $\log\sqrt{-1} = 0 \cdots$."

In a summary Euler stated (*l.c.*, p. 175):

"Les valeurs de	seront celles-cy à l'infini
$\dfrac{\log(+\sqrt{-1})}{\sqrt{-1}}$	$+\frac{1}{2}\pi;\ +\frac{5}{2}\pi;\ +\frac{9}{2}\pi;\ +\frac{13}{2}\pi;\ +\frac{17}{2}\pi;$ &c.
	$-\frac{3}{2}\pi;\ -\frac{7}{2}\pi;\ -\frac{11}{2}\pi;\ -\frac{15}{2}\pi;\ -\frac{19}{2}\pi;$ &c.
$\dfrac{\log(-\sqrt{-1})}{\sqrt{-1}}$	$+\frac{3}{2}\pi;\ +\frac{7}{2}\pi;\ +\frac{11}{2}\pi;\ +\frac{15}{2}\pi;\ +\frac{19}{2}\pi;$ &c.
	$-\frac{1}{2}\pi;\ -\frac{5}{2}\pi;\ -\frac{9}{2}\pi;\ -\frac{13}{2}\pi;\ -\frac{17}{2}\pi;$ &c."

[1] A discussion of his work in this connection may be found in F. Cajori's "History of logarithms" in this MONTHLY, *1913*, 75–84.

[2] *Mém. de l'acad. d. sc. de Berlin*, vol. 5 (1749), 1751, pp. 222–288. The memoir seems to have been presented to the Academy in 1746 (G. Eneström, *Verzeichnis der Schriften Leonard Eulers*, Erste Lieferung, 1910, p. 43).

[3] The proof that $(a + b\sqrt{-1})^{m+n\sqrt{-1}}$ is expressible in the form $x + y\sqrt{-1}$ is often referred to as "d'Alembert's theorem" [*e.g.*, *Annales de Mathématiques Pure et Appliquées* (Gergonne), July, 1913, p. 20]. He discussed the question in his *Reflexions sur la cause générale des vents*, Paris, 1747, p. 142, and in *Mém. de l'acad. de sc. de Berlin*, vol. 2 (1746), 1748, p. 192; cf. *Opuscules Mathématiques* . . . par M. d'Alembert, Paris, tome 1, 1761, p. 225 and tome 5, 1768, pp. 213–214. Part of this discussion was developed in L. A. de Bougainville, *Traité du calcul intégral, pour servir de suite à l'analyse des infiniment petits de l'Hôpital*, tome 1, Paris, 1754, pp. 42f. With Bernoulli, D'Alembert claimed that $\log\sqrt{-1} = 0$.

The "theorem" was also discussed by J. B. Labey in his notes to Euler, *Introduction à l'Analyse Infinitésimale*, tome 1, Paris, 1796, pp. 326–327; by Lagrange in his *Traité de la Résolution des Equations Numériques*, nouvelle édition, Paris, 1808, note IX; and by du Bourguet, and Gergonne, in *Annales de Mathématiques Pures et Appliquées*, tome 4, 1813, pp. 20–25. In *The Ladies' Diary*, 1833, p. 48, problem 1567 was, in effect: "When is $(a + b\sqrt{-1})^{m+n\sqrt{-1}}$ a real quantity? Determine whether all functions of $a + b\sqrt{-1}$ can be reduced to $A + B\sqrt{-1}$, or not." This was answered in the *Diary* for 1834, pp. 44–45. See also A. Cayley, *Proc. London Math. Soc.*, vol. 2, p. 54; *Coll. Math. Papers*, vol. 6, p. 68.

"Si $a = 0$; $m = 0$, & $b = 1$, il sera $c = 1$ & $\phi = \frac{1}{2}\pi$ d'où l'on tirera cette transformation:

$$(\sqrt{-1})^{n\sqrt{-1}} = e^{-2\lambda n\pi - \frac{1}{2}n\pi}$$

ou bien

$$(\sqrt{-1})^{\sqrt{-1}} = e^{-2\lambda\pi - \frac{1}{2}\pi},$$

qui est d'autant plus remarquable, qu'elle est réelle, & qu'elle renferme même une infinité de valeurs réelles differentes. Car posant $\lambda = 0$, on aura en nombres

$$(\sqrt{-1})^{\sqrt{-1}} = 0{,}2078795763507."$$

Euler gives a similar value in the last paragraph of a letter to Goldbach, dated June 14, 1746:[1] "Letztens habe gefunden, dass diese expressio $\sqrt{-1}^{\sqrt{-1}}$ einen valoren realem habe, welcher in fractionibus decimalibus = 0,2078795763, welches mir merkwurdig zu seyn scheinet."

But Euler's results were not generally known and accepted. For example, more than sixty-five years later we find Argand, notable for his geometrical interpretation of imaginary quantities,[2] stating that $\sqrt{-1}^{\sqrt{-1}}$ offers a simple example of a quantity which is irreducible to the form $x + y\sqrt{-1}$.[3] "He did not," as Hamilton remarks,[4] "anticipate De Morgan's theory of logometers."[5]

Among manuscripts published after Gauss's death (1855) were certain ones dealing with lemniscate functions. In one of these, values of $e^{-\pi}$, $e^{-(\pi/4)}$, $e^{-(9\pi/4)}$, $e^{\pi/2}$, and $11 \cdot e^{-(\pi/2)}$ are computed in an interesting manner (*Carl Friedrich Gauss Werke*, vol. 3, 1864, pp. 426–432). The values are as follows:

$e^{-\pi} = 0.0432139182\ 6377224977\ 4417737171\ 7280112757\ 2810981063,$

$e^{-(\pi/4)} = 0.4559381277\ 6599623676\ 5921294728\ 0294194166\ 0436523820,$

$e^{-(9\pi/4)} = 0.0008514383\ 4280515803\ 5852453295\ 4846487994\ 1872486024\ 8176915,$

$e^{\pi/2} = 4.8104773809\ 6535165547\ 3044648993\ 1536,$ and

$11 \cdot e^{-(\pi/2)} = 2.2866753398\ 58378$ which gives

$e^{-(\pi/2)} = .2078795763\ 50762$. On pages 418–419 (*l.c.*) Gauss sets down values for $2e^{-(\pi/4)}$ to 39 places of decimals, $2e^{-(9\pi/4)}$ to 27 places, $2e^{-\pi}$ to 40 places, $2e^{-(25\pi/4)}$ to 32 places, $2e^{-(49\pi/4)}$ to 28 places, $2e^{-4\pi}$ to 35 places, and $2e^{-9\pi}$, $2e^{-16\pi}$ to 27 places. The last five of these values are as follows:

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[1] *Correspondance Mathématique et Physique de quelques célèbres géomètres du XVIIIème siècle . . .* publiée . . . par P. H. Fuss. Tome 1, St. Pétersbourg, 1843, p. 383.

[2] *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques*, Pairs, 1806.

[3] *Annales de Mathématiques Pures et Appliquées*, tome 4, 1813, p. 146. Servois called attention to Euler's result (*l. c.* 1814, p. 231) in correcting this error. Details of the discussion in this connection are given in the volume of A. S. Hardy's English translation of Argand's *Essai* (Van Nostrand Science Series), New York, 1881.

[4] W. R. Hamilton, *Lectures on Quaternions*, Dublin, 1853, p. (56).

[5] See A. De Morgan, *Trigonometry and Double Algebra*, London, 1849, pp. 129–137; also R. B. Hayward, *The Algebra of Coplanar Vectors and Trigonometry*, London, 1892, pp. 119f.

$$2e^{-(25\pi/4)} = 0.0000000059\ 3851399312\ 9644497731\ 18$$

$$2e^{-(49\pi/4)} = 0.0000000000\ 0000003867\ 40505991$$

$$2e^{-4\pi} = 0.0000069746\ 8471241799\ 0983550387\ 96535$$

$$2e^{-9\pi} = 0.0000000000\ 0105109703\ 5201288$$

$$2e^{-16\pi} = 0.0000000000\ 0000000000\ 0295807$$

It will be observed that the values for $e^{-\pi}$ and $e^{-(\pi/2)}$ agree exactly with those given above by Professor Uhler, but that the values for $e^{\pi/2}$ differ—from the twenty-third decimal place on. Bastien's value, given below, appears to indicate that the value in Professor Uhler's paper is the correct one to twenty-eight places of decimals at least. By squaring Gauss's value for $e^{-(\pi/4)}$ Professor Uhler found the result to check to forty-seven places of decimals with his own value for $e^{-(\pi/2)}$.

Schellbach showed[1] in 1832 that many convergent series for π could be derived from such relations as

$$\pi = \frac{2}{i}\log i = \frac{2}{i}\log\frac{1+i}{1-i} = \frac{2}{i}\log\frac{(2+i)(3+i)}{(2-i)(3-i)} = \frac{2}{i}\log\frac{(5+i)^4(-239+i)}{(5-i)^4(-239-i)}$$

$$= \frac{2}{i}\log\frac{(10+i)^3(-515+i)^4(-239+i)}{(10-i)^3(-515-i)^4(-239-i)}.$$

Benjamin Peirce referred,[2] in 1882, to "the mysterious formula"

$$i^{-i} = e^{\pi/2} = 4.810477381.$$

The phrase "l'équation symbolique et mystérieuse

$$\frac{\pi}{2}\sqrt{-1} = \text{Log.}\,(\sqrt{-1})"$$

was employed by J. F. Francais[3] in September, 1813.

In the introduction to *Table d'Interpolation pour le calcul des parties proportionnelles faisant suite* aux *Tables de Logarithmes* . . . par L. Schrön precédé d'une introduction française par J. Houel (Paris, 1891) the following number is evaluated

$$e^{-\pi} = 0.043213918263772248\cdots.$$

In 1919, Brocard suggested[4] that "la connaisance des nombres e^{π} and π^e donnerait peut-être l'indice de quelque relation entre e et π." E. Chanzy found such values[5] to be $e^{\pi} = 23.1406926327787\cdots$, and $\pi^e = 22.4591577183\cdots$;

[1] *Journal für die reine und angewandte Mathematik*, vol. 9, pp. 404–405.
[2] B. Peirce, *Linear Associative Algebra*, New York, 1882, p. 5.
[3] *Annales de Mathématiques Pures et Appliquées*, vol. 4, p. 67.
[4] *L'Intermédiaire des Mathématiciens*, vol. 26, p. 73, question 4935.
[5] *Sphinx-Oedipe*, August, 1920, pp. 127–128.

and L. Bastien, with the aid of his tables of logarithms to 32 places of decimals found[1]

$$e^{\pi} = 23.14069263277926900572908636\underline{7}9,$$

$$e^{\pi/2} = 4.81047738096535165547303566\underline{6}7,$$

$$\pi^{e} = 22.45915771836104547342715220\underline{4}5.$$

The symbol i for $\sqrt{-1}$ was first used by Euler in a "M.S. Academiæ exhibit. die 5 Maii 1777"[2] printed posthumously in 1794.[3] This notation was adopted by Gauss in 1801.[4]

The symbol π for the ratio of the circumference of a circle to its diameter was first used by W. Jones in 1706.[5] It was probably suggested to Jones by Oughtred who employed the symbol in a different sense.[6] Euler's first use of the symbol was in 1737[7]; up to that time he had used the letter p.

The symbol e for the number, defined by the series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}\cdots$, was first used by Euler in 1731.[8]

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

2. Dupin as Secretary of the Ionian Academy.

Among the most interesting but by no means most scholarly of the French mathematicians of the first part of the nineteenth century was that mélange of economist, politician, geometrician, and popularizer of science, Pierre-Charles-François Dupin (1784–1873). To the mathematician he is chiefly known as the favorite pupil of Monge and as his biographer; for his *Développement de géométrie*

[1] *L'Intermediaire des Mathematiciens*, vol. 27, p. 65, May–June (not published till October), 1920. In the value for π^e given above the number "1" has been inserted in the seventh place from the end where "r" appears in the original.

[2] L. Euler, "De formulis differentialibus angularibus maxime irrationalibus, quas tamen per logarithmos et arcus circulares integrare licet."

[3] L. Euler, *Institutiones calculi integralis*, vol. 4, 1794, pp. 183–184. The symbol i is introduced on p. 184.

[4] C. F. Gauss, *Disquisitiones arithmeticae*, 1801, p. 337; *Werke*, vol. 1, 1870, p. 414.

[5] W. Jones, *Synopsis Palmiorum Matheseos*, London, 1706, p. 263.

[6] William Oughtred (1574–1660) in his *Clavis Mathematica* of 1647, etc., and in his *Theorematum in libris Archimedes de Sphaera et Cylindro Declaratio*, Oxford, 1652, frequently employs the symbol $\delta : \pi$ or $\pi : \delta$ (in modern notation) for the ratio of the semi-diameter to the semi-periphery or of semi-periphery to semi-diameter. It is noticeable that these letters are *never* used separately, that is, π is *not* used for "Semiperipheria," as Tropfke suggests (*Geschichte der Elementar-Mathematik*, vol. 2, 1903, p. 135). Oughtred states specifically in his "Theorematum": "$\frac{\pi}{\delta}R$, est semiperipheria circuli cujus Radius est R." In 1697 David Gregory used (*Philosophical Transactions*, vol. 19, p. 652) π/ρ to designate the ratio of the circumference to the diameter.

[7] L. Euler, "Variae observationes circa series infinitas," *Comment. acad. sc. Petrop.*, vol. 9 (1737), 1744, p. 165.

[8] *Corresp. math. et phys* . . . par Fuss, vol. 1, 1843, p. 58: "e denotat hic numerum, cujus logarithmus hyperbolicus est = 1."

(Paris, 1813); and for his contributions to the theory of curvature of surfaces, and the theory of light, the study of mechanics, curves of the third and fifth orders, and conjugate tangents. After having made a high record at the École polytechnique he entered the navy. His appointment to study the naval establishment of Great Britain led to the publication of his *Voyages dans la Grande-Bretagne* (Paris, 6 vols., 1820–1824). In 1819 he became professor of mechanics at the Conservatoire des Arts et Métiers, in 1824 he was created a baron by Louis XVIII, in 1837 he was made a peer, and in 1840 he became grand officer of the Legion of Honor. In the popular brilliancy of his career he showed much of the ability of his mother. It was largely she who directed the education of her three sons, all of whom became men of prominence, and it speaks well for her appreciation of their talents that she had placed upon her tomb the words, "Ci-git la mère des trois Dupin." It was with reference to this inscription that Charles, who survived his brothers, was called in later life "the last of the Dupins."

Of the several letters of Dupin now in my collection perhaps the one that shows most clearly the great diversity of his interests relates to an incident in his life not generally known to those who study his contributions to geometry.

The letter to which I refer calls attention to quite a different line of interest and shows Dupin at the age of twenty-five in a capacity quite foreign to his mathematical interests. It reads as follows:

CORCYRE (CORFU), September 26, 1809; the 647th Olympiad, 2d year.
ACADEMY.

Μετὰ Θάνατον ἀνίσαται

Ch. Dupin, Secretary of the Ionian Academy for the French language and the official correspondence, Engineer in the French Navy,

To Monsieur De Gérando, Member of the Consulta charged with organizing the Roman states, and Secretary of the National Institute of France, 3d Classe.

Sir: I have the honor of laying the program of the Prix Olympiadiques before the Secretary of the Institute, in the Classe which has antiquity for the special subject of its meditations. Nothing which relates to Greece can be foreign to you.

We present to you our homage . . . . You belong to the Consulta, created for the regeneration of Rome and Latium. We believe that we see in you the representative of the sciences and arts of France, sent to modern Rome for the purpose of repaying with interest the benefits which Europe received from ancient Rome.

Finally, Sir, it is pleasing to me that, as the representative of a learned society, I am allowed to pay my respects to an old friend of my father's (Dupin, de la Nièvre, ex-legislator, Rue du bacy, No. 998).

I earnestly hope that the program of our prizes may be so fortunate as to command your approval, and I could wish nothing more than that you would accede to the request which I have the honor to make.

Permit us to add a notable name to the list of associates who honor us, and among whom we are able to name Monti, Fabroni, Chaptal, and many other members of the Institute of France.

How great would be my happiness if I could be able to present to the Ionian Academy a present such as that of your fellowship in this institution, young and still feeble, but which recalls at least some of the glorious ideas of ancient Greece.

I have the honor, Sir, of saluting you with the highest consideration.

CH. DUPIN.

The letter thus presents a single interesting incident in a very interesting life,—a life that was probably more extended than that of the Academy which he was influential in establishing.

3. Picard[1] and Cassini.

Scientific letters more than two hundred years old are becoming rare. They exist in the museums and great libraries, particularly in Europe, but every year it is becoming more difficult to secure them for private collections. One of the most valuable, of those which I have, came into my possession recently, and is of particular interest because of the men concerned and because it contains the results of certain important astronomical observations. The letter was written at Brest on October 2, 1679, by Jean Picard, and is addressed to "Monsieur Cassini, a l'observatoire Royal au faus-bourg St. Jaques. A Paris." Picard was then fifty-nine years of age, and this letter was written about three years before his death. His life had, as is well known, been actively devoted to scientific work. He was born in 1620 and became the foremost astronomer of his time, greatly improving the instruments of observation, making a very satisfactory measurement of a degree of a meridian (1669), and writing on the form of the earth. It was his measurements which enabled Newton to confirm his theory with respect to gravitation.

The Cassini to whom the letter was written was Giovanni Domenico (Jean-Dominique). He was for a time professor of astronomy at Bologna, but in 1669 the pope agreed to his acceptance of a call to Paris, where he became the first of four generations to fill the post of director of the great French observatory. He determined the motion of Jupiter's satellites, discovered four of Saturn's, made a close approximation to the parallax of the sun, corrected Kepler's value of the earth's eccentricity, and first described the ovals that bear his name.

It was very likely in connection with Cassini's observations on the satellites of Saturn that Picard, wishing to assist him, wrote the following letter:

> "I have already written you that, on the twenty-fourth, at 16 H. 15′ 56″ we observed the immersion of the first satellite in the shadow; this observation, which was very favorable, gives a difference of 27′ 33″ in comparison with the one that you made on the night of the twenty-sixth . . . I also made this last observation, but did not consider it valuable because of the humidity at that time. . . . We were more fortunate on the evening of the twenty-ninth, when the immersion of the second satellite took place at precisely 10 H. 16′ 47″. . . . We expect to leave on Tuesday, Oct. 4, or at the latest on Thursday, wishing to observe once more the immersion of the first satellite which takes place the night of the 3–4th.

The letter then proceeds to discuss more personal matters, relating in part to asking Cassini to write to "Mr. l'Abbé" about their departure. This M. l'Abbé was evidently an older scientist, for Picard remarks that he had been "pretending to make some dissections of fish, but an old man like him ought to be content with dissecting sardines,"—which shows that the greatest savants have their sense of humor.

[1] See the article in *Revue générale des Sciences* referred to on page 137 of this issue of the Monthly.—Editor.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

A number of questions have been standing in this department for some time with no replies or with replies which still leave something to be desired in the way of completeness or finality. Several of these are reprinted below, with the intention of directing our readers' interest to them again, and stimulating replies. To those which have already received some attention short notes are appended, indicating the extent to which they still remain open for consideration.

QUESTIONS.

15 [*1914*, 278; *1916*, 353; *1919*, 68; *1920*, 114, 361]. In the *Proceedings of the Royal Society of Edinburgh*, vol. 7, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

"If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube."

How does this "easy proof" follow?

A large number of replies to this question have been received; they are analyzed in the remarks by the editor, *1920*, 361. It seems necessary to emphasize again that what is desired is *not* a proof that if $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$—this is of course very easy; but a proof, by use of this theorem, that no two positive integers exist the sum of whose cubes is a cube.

21 [*1914*, 341; *1916*, 354; *1919*, 68, 239; *1920*, 114]. For the Diophantine equation

$$x^2 - y^3 = 17$$

there are known the following solutions:

$$\begin{array}{lrrrrrrrr} x = & 3, & 4, & 5, & 9, & 23, & 282, & 375, & 378661, \\ y = & -2, & -1, & 2, & 4, & 8, & 43, & 52, & 5234. \end{array}$$

One of our readers, who supplied the foregoing facts, desires to know the answers to the following questions: Are there other solutions of the given Diophantine equation? How may all the solutions of this equation be found by a systematic procedure?

In a reply to this question published in the MONTHLY for June, 1919, E. B. Escott showed how to obtain an infinite number of rational solutions of the equation, and in particular all the integral solutions contained in the above list.

A number of references bearing on the question are given by Dickson, *History of the Theory of Numbers*, vol. 2, chapter 20, pp. 533–539. Of especial pertinence are the following:

Mordell, *Proceedings of the London Mathematical Society*, series 2, vol. 13, pp. 60–80; vol. 18, pp. v–vi.

Thue, *Crelle's Journal*, vol. 135, pp. 303–304.

30 [*1916*, 88, 354; *1920*, 114, 362]. A certain Normal University wishes to offer thirty-five hours of college mathematics for the benefit of high-school teachers. What should these courses

be in order that, primarily, they may be of the greatest value to high-school teachers of mathematics and, secondarily, that they may furnish stimulus for a more extended pursuit of the subject?

Only one reply to this question has been received, *1920*, 362. It seems that more might be said on this question, and further replies are desired. Some idea of existing conditions may be obtained from the replies to another question in the MONTHLY, *1916*, 395.

34 [*1917*, 134, 341; *1920*, 114, 301, 405, 460; *1921*, 19]. Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

There has been a considerable amount of interesting discussion on this question, but its possibilities are by no means exhausted. A formulation of the sense in which the problem is to be interpreted was given by the editor, *1920*, 302. For continuous functions, analytic at the origin, the only solutions are cubic polynomials. It was shown by Gillespie, *1920*, 405, that the condition of analyticity at the origin may be replaced by the existence of six continuous derivatives near the origin. On the other hand remarks by Bennett, *1920*, 460, Weisner, *1921*, 20, and the editor, *1920*, 462, show that there exist solutions other than cubic polynomials, possessing continuous first derivatives. It would be of interest to know how little need be assumed beyond the continuity of the first derivative to restrict solutions to cubic polynomials.

The related question obtained by taking both limits of integration variable and suitably modifying the formula has reached a satisfactory answer with the proof by Ballantine, *1921*, 19, that the solution must be a cubic polynomial provided it is merely continuous.

35 [*1918*, 266; *1920*, 114]. Is the following theorem new or has it previously been published?

THEOREM. *If two parallel planes, π and π', cut sections from a cylindrical surface S and two spherical surfaces S_1 and S_2, and if the sum of the sections of S_2 is equal in area to the sum of the sections of S plus the sections of S_1, then the part of S_2 included between π and π' is equal in volume to the sum of the parts of S and S_1 included between π and π'.*

In communicating this question, E. O. Brower commented on the simplicity with which the theorem could be proved, which led him to wonder whether so simple a proposition had remained unnoticed.

39 [*1920*, 256]. There are certain problems in geometry which are simple in statement but can be reduced only to very complicated problems in transcendental analysis. Following are several examples of the type of problem in question.

1. What is the smallest plane area within which a given figure can be turned through a complete revolution? It is not implied that the figure should revolve about a fixed point, but merely that in the course of its motion the figure should have every possible orientation in the plane. The problem may be modified by considering only convex areas.

An interesting special case is that in which the given figure is a segment of a straight line. In this case it has been conjectured by Professors Osgood and Kubota that the smallest area may be bounded by a three-cusped hypocycloid; if we consider only convex areas, perhaps the result will be an equilateral triangle. I have no indication of a proof.

2. For every closed convex curve of area P there is an n-sided circumscribed polygon of least area Q and an n-sided inscribed polygon of greatest area R. For a fixed value of the integer n and for all closed convex curves, what is the upper limit of Q/P and what is the lower limit of R/P? I have succeeded only in proving that for the case $n = 3$, the upper limit of Q/P is 2.

3. Let the area of a given simple closed curve A be a. Remove from A the greatest possible area a_1 similar to another given simple closed curve B. From the remaining figure remove the greatest possible area a_2 similar to B. Continue this process indefinitely. Is it or is it not true that

$$a_1 + a_2 + a_3 + \cdots = a?$$

I have proved the statement to be true in the special case that A is convex and B is a circle.

4. Let a given closed convex curve K have the property that a given triangle whose angles are incommensurable with n can be revolved completely within K (see Part 1 of this question), always remaining inscribed in K. What may the curve K be? Can any other curve except a circle satisfy the conditions?

These questions were formulated by Professor S. Kakeya, who has treated some problems of similar nature in various papers in the *Science Reports* of the Tôhoku Imperial University. They involve considerations of decided difficulty in the analytic formulation of geometric relations. The special case mentioned in the first question has received some attention from W. B. Ford, *Bulletin of the American Mathematical Society*, vol. 27, p. 55.

40 [*1920*, 365]. How great emphasis is laid in freshman mathematics upon the elementary algebra of complex numbers? A recent paper by an eminent electrical engineer seems to indicate the need of a knowledge of this subject on the part of draftsmen and mechanics of very limited educational opportunities. The syllabus of the College Entrance Examination Board mentions the topic under the caption "Advanced Angebra," but the question papers call for only the slightest study of numbers of this type. Is Euler's theorem ($e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$) usually presented in college algebra, or is it left to the calculus? Does the topic deserve greater emphasis than it usually gets, for the sake of applications in the field of periodic currents?

The discussion, "Complex Numbers in Advanced Algebra," by Mr. H. E. Webb, *1920*, 411, may be viewed as a reply to this question. There seems to be room for further discussion.

41 [*1920*, 365]. A reader asks for an elementary proof of the following two propositions in number theory, either of which can readily be obtained from the other:

Every positive integer of the form $8n + 3$ is the sum of three odd squares.

Every positive integer is the sum of not more than three triangular numbers.

Bachmann[1] states that these theorems have been proved only by the use of the theory of ternary quadratic forms, and refers to the discovery of the theorems by Gauss, and a comparatively simple proof by Dirichlet,[2] by means of this theory. A number of references are given by Dickson.[3] Apparently the theorem was first enunciated by Fermat.

42 [*1921*, 65]. In connection with the questions of Kakeya, Professor W. B. Ford is led to the following inquiry: A line-segment AB is to be moved in its plane to a new position $A'B'$. How should this be done in order that the area generated may, to the greatest extent possible, be passed over three times?

The questions of Kakeya here referred to are given as Question 39 above.

[1] *Niedere Zahlentheorie*, Leipsic and Berlin, 1910, Teil 2, p. 325.

[2] *Crelle's Journal*, vol. 40, p. 228; *Liouville's Journal*, series 2, vol. 4, p. 233.

[3] *History of the Theory of Numbers*, vol. 2, Chapter 1.

Professor Ford has proved that if the generated area is to be passed over, to the greatest possible extent, but *two* times, AB should be rotated about the intersection of the perpendicular bisectors of AB and $A'B'$.

DISCUSSIONS.

Doctor Morris gives a device for a perpetual calendar, with an account of the theory on which it is constructed. A similar explanation will hold for any of the many forms of such calendars. It is believed that this article will be of interest, since explanations of the theory of perpetual calendars are seldom given.

Professor Bennett gives a set of simple identities from which a four-place logarithm table can readily be calculated. The method is probably new. It bears some relation to the plan used by Briggs (cf. *Encyclopædia Britannica*, 11th ed., vol. 16, pp. 875–876); but Briggs needed first to calculate and use explicitly the modulus, while Professor Bennett does not make any use of the modulus.

I. The Theory of Perpetual Calendars.

By Frank R. Morris, University of California.

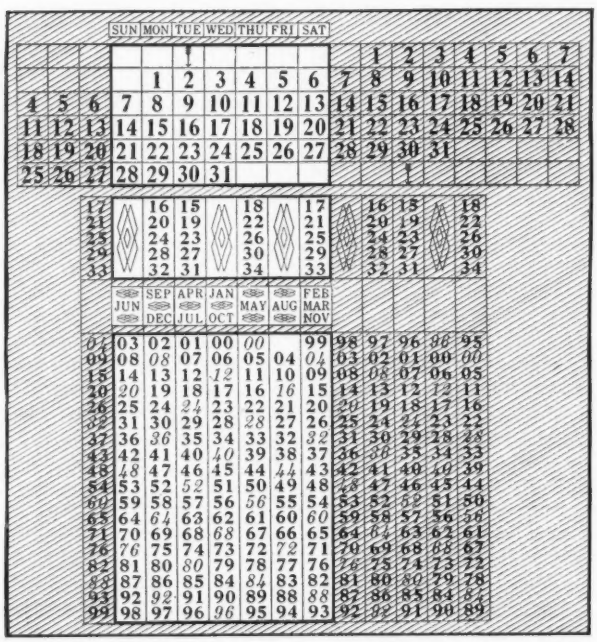
A short discussion of adjustable calendars is given by Irwin Roman[1] in the Monthly, *1915*, 241. He describes one of the several hundred mechanical devices used in presenting the calendar of a given month of a given year, but he does not give the mathematical theory upon which practically all perpetual calendars are based. It is the purpose of this article to present this theory.

There are approximately 365.2422 days in a tropical year. If each calendar year contained 365 days, in the course of 400 years an error of 96.88 days would accumulate. Then if 97 days be added for each period of 400 years the error will be very small. In any 400 consecutive integers there are 100 numbers which are divisible by 4, and 3 which are divisible by 100 but not by 400. Hence the rule: *Each year which is divisible by* 4 *but not by* 100 *and each year which is divisible by* 400 *is a leap year with* 366 *days. All other years contain* 365 *days.* Since this rule was first introduced by Pope Gregory XIII in the year 1582, no calendar based upon it need antedate the sixteenth century. The calendar may be extended into the future as far as one may choose. However, it becomes inaccurate after a few thousand years, because it accumulates a day in about 3,000 years, and also because the tropical year varies slightly as the years go by. Other calendars, which are used in some parts of the world or have been used in the past, are built upon other rules but I shall discuss only that calendar

[1] Mr. Roman's article begins with the statement, "So far as the writer has been able to learn, all perpetual or adjustable calendars are arranged so as to present the first day of the month as the first day of the week." He had evidently overlooked the most fruitful field for the study of perpetual calendars, viz., the *Official Gazette* of the United States Patent Office. During the past 25 years more than 100 calendars have been patented and more than a score of these present the days of the week in the natural order. For example see number 1048413, Dec. 24, 1912. [Doctor Morris's perpetual calendar was patented Nov. 26, 1918, number 1286058.—Editor.]

which results from the assumption of the above rule. The same general principles are used even though the rules are different.

Most perpetual calendars covering a period of several centuries contain five parts. These parts are *the days of the week, the days of the month, the months of the year, the years of the century* and *the centuries*. The days of the week are placed in regular order, beginning with any one of the seven days, and form an endless set of cycles of seven days each. The days of the month are the integers from 1 to 31 and are arranged in the usual manner. The months form an endless set of cycles of twelve months each. The years of the century, which occur in cycles of 100 years each, are indicated by the two right-hand digits of the numbers of the years; and the centuries are indicated by the numbers of the years after the years of the century have been displaced. They are the endless chain of integers which in our case begin with 15. The number which indicates the century is one less than its ordinal number. In the figure given herewith these five parts are shown from top to bottom in the order—days of the week, days of the month, centuries, months of the year, years of the century.



Let us start with the calendar for January, 1900, which is shown in the figure. Monday is the first day of the month. Since January has 31 days and $31 \equiv 3 \pmod 7$, *i.e.*, if 31 is divided by 7 the remainder is 3, February enters 3 days later in the cycle of days of the week than did January, which is Thursday. Hence the days of the week must be moved to the left 3 spaces or the days of the months to the right 3 spaces. Let us assume the latter movement and that the days of the week at all times remain in a fixed position. Also assume for the present that 00 is on a card which moves with the days of the month. It may be placed in any one of the seven columns. In the same column, but on the fixed card which contains the days of the week, place the month January. Now when the days of

the month are moved to the right 3 spaces 00 will be affected in the same manner. Then place February in the column containing the new position of 00, which is 3 spaces to the right of January. According to our rule February of 1900 contains 28 days. Thus the first day of March also falls on Thursday and March is in the same column as February. Since March has 31 days April is placed 3 columns to the right of March in the cycle of 7, which is one column to the left of January. This process may be continued until all the months of the year are located on the fixed card. When this is done the calendar for any month of the year 1900 is given when 00 is placed in the column of the desired month. The calendar shows 31 days for each month, but it is understood that only the proper number is to be used.

Now consider the year 1901. There are 365 days in 1900, and $365 \equiv 1 \pmod 7$. Hence every day of the year 1901 falls one day later in the week than the corresponding day of the year 1900, and 01 should be placed one space to the left of 00, causing the days of the months to be shifted one space to the right. When 01 is moved to the column of any month the calendar is given for that month in 1901. Likewise 02 is located one space to the left of 01, and 03 one to the left of 02. However, the extra day of 1904, which is a leap year, must be accounted for. January and February run regularly, but the remaining 10 months of the year enter *two* days later in the week than they did in 1903. Hence we must have two symbols for a leap year. Let the italic figures *04* placed one space to the left of 03, which is in the cycle 3 spaces to the right of 00, be used for January and February, and let the usual figures 04 be placed two spaces to the left of 03 and used for the remaining 10 months of the year. Similarly the rest of the years of the century can be located.

We still have to consider the shift which must be made in passing from one century to the next. There are 36525 days from March 1, 1900 to March 1, 2000; and $36525 \equiv 6 \pmod 7$. With the exception of the first two months, the months of the 21st century begin six days later in the week than did the corresponding months of the corresponding years of the 20th century. Hence in changing from 1999 to 2000 the days of the month must be moved 6 spaces to the right with respect to the years of the century. This can be accomplished if the days of the month are on an auxiliary slide which is mounted upon the main slide containing the years of the century. Now place 19 in any one of the columns on the main slide and 20 in the column 6 spaces to the right of 19. Before the auxiliary slide is moved from its original position place an arrow on it in the column containing 19. Then when the auxiliary slide is moved so that the arrow points to 20 the days of the month and the years of the century are in the correct relative position for all years of the 21st century. The first two months of the century are cared for by placing *00* in italics one space to the right of 00. There are 36524 days from March 1, 2000, to March 1, 2100; and $36524 \equiv 5 \pmod 7$. Hence the days of the month must be shifted 5 spaces to the right relative to the years of the century, which means that 21 should be 5 spaces to the right of 20 on the main slide. Likewise 22 should be 5 spaces to the right of 21, and 23

five to the right of 22; but 24 should be 6 spaces to the right of 23. The shift is 6 spaces for all centuries divisible by 4 and 5 spaces for the other centuries. This may be continued to any desired extent and may be extended backwards to the beginning of the calendar.

It is easy to see from the figure that there must be at least 13 columns on the main slide of this particular form of the calendar. With the given position of the centuries it is necessary to have 18 columns on the auxiliary slide. This number might have been reduced by one; but there would then have been a loss in symmetry in two parts. Other forms of the calendar allow different parts to move. A common type has the parts on disks, which rotate with respect to one another. Still another type uses one or more of the parts as reference tables.[1] However, the theory which I have given is applicable to practically all types.

II. Note on the Computation of Logarithms.

By Albert A. Bennett, University of Texas.

The following approximate relations are easily verified by the use of a computing machine or by direct multiplication with paper and pencil, and without recourse to logarithms. (The writer checked these on a "Monroe" in about two hours.)

(1) $(1.024)^{214} = 160.02580 \cdots$,
(2) $3\,(9/8)^{18} = 24.99577 \cdots$,
(3) $2 \times 5 = 10$,
(4) $7\,(98/81)^{4} = 14.998996 \cdots$,
(5) $((2.2)^{3}\,(2.1))^{2} = 500.005377 \cdots$,
(6) $39\,(13/14)^{6} = 25.000945 \cdots$,
(7) $25\,(17/14)^{5} = 66.000071 \cdots$,
(8) $20\,(190/33)^{2} = 662.9936 \cdots$,
(9) $13\,(11/10)^{2}\,(17/13)^{4} = 45.9992752 \cdots$,
(10) $551\,(55/221)^{2}\,(23/39)^{3} = 6.9997431 \cdots$.

If these relations be replaced by the approximate relations

(1′) $(1.024)^{214} = 160$,
(2′) $3\,(9/8)^{18} = 25$,
(3′) $2 \times 5 = 10$,
(4′) $7\,(98/81)^{4} = 15$,
(5′) $((2.2)^{3}\,(2.1))^{2} = 500$,
(6′) $39\,(13/14)^{6} = 25$,
(7′) $25\,(17/14)^{5} = 66$,
(8′) $20\,(190/33)^{2} = 663$,
(9′) $13\,(11/10)^{2}\,(17/13)^{4} = 46$,
(10′) $551\,(55/221)^{2}\,(23/39)^{3} = 7$,

[1] A very interesting example of this type is given by Augustus De Morgan in a volume of 88 pages entitled *The Book of Almanacs*. In addition to what is given by the ordinary perpetual calendar, his book includes saints' days, lunar calendars, several special calendars and discussions of historical interest.

and the logarithms to the base 10 be found formally, these reduce respectively, to the following:

$$(1'')\quad 2136 \log 2 = 643,$$
$$(2'')\quad 37 \log 3 = 2 + 52 \log 2,$$
$$(3'')\quad \log 5 = 1 - \log 2,$$
$$(4'')\quad 9 \log 7 = 1 - 5 \log 2 + 17 \log 3,$$
$$(5'')\quad 6 \log 11 = 11 - 7 \log 2 - 2 \log 3 - 2 \log 7,$$
$$(6'')\quad 7 \log 13 = 2 + 4 \log 2 - \log 3 + 6 \log 7,$$
$$(7'')\quad 5 \log 17 = -2 + 8 \log 2 + \log 3 + 5 \log 7 + \log 11,$$
$$(8'')\quad 2 \log 19 = -3 - \log 2 + 3 \log 3 + 2 \log 11 + \log 13 + \log 17,$$
$$(9'')\quad \log 23 = -2 - \log 2 + 2 \log 11 - 3 \log 13 + 4 \log 17,$$
$$(10'')\quad \log 29 = -2 + 2 \log 2 + 3 \log 3 + \log 7 - 2 \log 11 + 5 \log 13 + 2 \log 17 - \log 19 - 3 \log 23.$$

These relations may be used as recursion equations to determine approximately the logarithms of the first few primes. When solved successively it is found that log 2 is given correctly to six places, so that its powers up to 1024 are correct to five places, and each of the other primes has its logarithm given correctly to at least five decimal figures.

These relations are found to be ample for completing correctly the construction of a four-place logarithm table, by use of linear interpolation when once the logarithms of all numbers of three significant figures or less, obtainable as powers and products of the above primes, have been written down. For example, between the logarithm of 23 and the logarithm of 29, we have without interpolation and by mere use of the fact that the logarithm of a product is equal to the sum of the logarithms of the factors, the logarithms of 24, 25, 26, 27, 28, as well as of such numbers, for example as, 25.6. With a little ingenuity (such as would be illustrated in determining the logarithm of 7, from the logarithms of 2, 3, and 5, by the relation that 49, lies between 48 and 50), a five-figure table with at most very few errors can be obtained from the above relations.

These relations have been found by an extension of the continued fraction algorithm, and for the accuracy desired simpler relations do not appear to exist.

This alternative method of constructing or checking logarithm tables is believed to be new.

RECENT PUBLICATIONS.

REVIEWS.

A First Course in Nomography. (Bell's Mathematical Series.) By S. Brodetsky. London, G. Bell and Sons, 1920. 12 + 135 pages. Price 10 shillings.

Preface: "In many branches of science, in engineering practice, in technology, in industry and in military science, Nomography is a recognized means of carrying out graphical calculations. The ballistic constant in gunnery, flame temperature in the research of coal-gas combustion, the angle of twist in a thread of given thickness with a given number of turns per inch, the conversion of counts in the textile industry, can all be calculated by means of nomograms. Nomographic

charts are simple and certain in use, so that calculations formerly entrusted to skilled and responsible computers can now be safely left to the care of a comparatively unskilled subordinate. It is the object of this First Course to offer a clear and elementary account of the construction and use of such charts.

"The method of treatment chosen is based on experience gained in the making of nomograms for various technological departments in the University of Leeds, and in other ways. It is a treatment that should be found useful by the reader who desires to become acquainted both with the theory of nomography and with its practical use. Chapter III begins the nomography proper, but the reader is advised to study Chapters I and II first, in order to see how the nomograms in Chapter III can be constructed. Special attention is directed to §§ 49–50 in Ch. IV, and to Chapter VIII. Answers have been purposely omitted, even where the examples lead to numerical results. . . ."

Contents: List of diagrams and index of nomograms, ix–x; Historical sketch, xi–xii ["When Descartes invented Coördinate Geometry, he put at the disposal of mathematicians a powerful weapon that has led to phenomenal advances in all branches of mathematical science. For the purpose of practical use in calculations by means of graphical representation on a plane, the number of variables that can be used is obviously restricted to two. This limitation was removed by Buache (1752), who introduced the method of contours—now incorporated in all atlases and surveys. By means of contours it is possible to deal with three quantities at once, whilst they are all represented on one plane, as, *e.g.*, in indicating the variation in the height of land or the depth of the sea. This sufficed for a time, but the extraordinary growth of railway systems all over the world led to important developments by Lalanne (1841), Massau (1884), and Lallemand (1886).

"The idea of using collinear points, which constitutes the chief beauty of the method of the present book, was developed by d'Ocagne (1884). It was d'Ocagne, too, who applied the name Nomography to this method, in his book *Les calculs usuels effectués au moyen des abaques* (1891). Since then further extensions have been made by d'Ocagne and others.

"In recent years the utility and convenience of nomography have been increasingly realized, and the subject has gained in importance and recognition, particularly in engineering practice. It is, in the main, a product of French mathematical genius. Articles have appeared in one or two English journals, and excellent accounts of the subject in English are to be found in Hezlett's (*sic*) *Nomography* (Royal Military Institution, Woolwich) and Lipka's *Graphical and Mechanical Computation* (Wiley, New York).

"But the reader who is interested in the subject cannot do better than read d'Ocagne's excellent *Traité de Nomographie* (G. Villars, Paris, 1899)"]; Introduction, 1–5; Chapter I: Nomograms for addition and simultaneous equations, 6–22; II: Generalized nomograms for addition and subtraction, 23–45; III: Nomograms for multiplication and division, 46–64; IV: Nomogram with two parallel scales. Quadratic equations, etc., 65–88; V: Generalized theory of nomograms with two parallel scales. Parallel coordinates, 89–99; VI: Nomograms with trigonometrical functions, 100–108; VII: Nomograms with intersecting scales, 109–117; VII: Practical and automatic construction of nomograms. Empirical nomograms. Practical details, 118–135.

Leçons sur l'intégration des équations aux dérivées partielles du premier ordre. Par EDOUARD GOURSAT. Deuxième édition revue et augmentée. Paris, Hermann, 1921 [published October, 1920]. 2 + 459 pages. Price 40 francs.

The first edition of this work appeared in 1891. In the present edition considerable change has taken place and the work as a whole is nearly one third larger. Goursat remarks in the preface:

"Dans cette nouvelle édition, on a conservé le plan général de ces Leçons. Seuls les premiers Chapitres, relatifs aux théorèmes d'existence et aux équations linéaires, ont été assez profondément remaniés. Dans la suite, on a simplement modifié quelques demonstrations et complété certains résultats.

C'est aussi pour ne pas bouleverser l'ordre suivi dans l'édition originale que je n'ai pas introduit dans ce Volume la méthode de Pfaff. Cette importante méthode sera exposée, avec les développements qu'elle mérite, dans un autre Ouvrage, spécialement consacré au *Problème de Pfaff* et à ses généralisations, et qui paraîtra, je l'espère, prochainement." The publisher expects that this new volume will be ready in September, 1921.

Contents—Chapter I: Théorèmes d'existence, 1–49; II: Equations linéaires, Systèmes complets, 50–102; III: Equations linéaires aux différentielles totales, 103–133; IV: Intégrals complètes. Méthode de LaGrange et Charpit, 134–161; V: Méthode de Cauchy. Caractéristiques, 162–201; VI: Etude géométrique des équations a trois variables. Courbes intégrales. Solutions singulières, 202–253; VII: Première méthode de Jacobi, 254–266; VIII: Seconde méthode de Jacobi. Généralisations de Mayer et de Lie, 267–308; IX: Théorie générale de Lie, 309–352; X: Transformations de contact, 353–409; XI: Groupes de fonctions. Méthode générale d'intégration, 410–454.

Pioneers of Progress: *Archimedes*. By T. L. HEATH. ("Men of Science" series edited by S. Chapman). London, Society for Promoting Christian Knowledge, 1920. 2 + 58 pages. Cloth. Price 2 shillings.

First two paragraphs: "If the ordinary person were asked to say off-hand what he knew of Archimedes, he would probably, at the most, be able to quote one or other of the well-known stories about him: how, after discovering the solution of some problem in the bath, he was so overjoyed that he ran naked to his house, shouting εὕρηκα, εὕρηκα (or, as we might say, 'I've got it, I've got it'); or how he said 'Give me a place to stand on and I will move the earth'; or again how he was killed, at the capture of Syracuse in the Second Punic War, by a Roman soldier who resented being told to get away from a diagram drawn on the ground which he was studying.

"And it is to be feared that few who are not experts in the history of mathematics have any acquaintance with the details of the original discoveries in mathematics of the greatest mathematician of antiquity, perhaps the greatest mathematical genius that the world has ever seen."

Contents—Chapter I: Archimedes, 1–6; II: Greek geometry to Archimedes, 7–23; III: The works of Archimedes, 24–28; IV: Geometry in Archimedes, 29–44; V: The sandreckoner, 45–49; VI: Mechanics, 50–52; VII: Hydrostatics, 53–56; Bibliography, 57; Chronology, 58.

An Introduction to String Figures. An Amusement for Everybody. By W. W. R. BALL. Cambridge, W. Heffer & Sons, 1920. 38 pages. Price 2 shillings.

"Prefatory Note" dated July, 1920: "The making of String Figures is a game common among primitive people. Its study by men of science is a recent development, their researches have, however, already justified its description as a hobby, fascinating to most people and readily mastered. The following pages contain a lecture on these figures and their history; to it I have appended full directions for the construction of several easy typical designs, arranged roughly in order of difficulty, and, for those who wish to go further, lists of additional patterns and references. The only expense necessary to anyone who takes up the pastime is the acquisition of a piece of good string some seven feet long; with that and this booklet to aid him, he will have at his command an amusement that may while away many a vacant hour."

Newton. (Profile N. 52). By GINO LORIA, Roma, A. F. Formiggini, 1920. 69 pages. Price 3.00 lire.

This is the latest volume in the dainty little series of booklets (4 x 6½ inches) among which A. Mieli's *Lavoisier* (no. 42), A. Favaro's *Archimedes* (no. 21), and *Galileo* (no. 10) have been published during the past ten years. Each volume is written with light touch by one thoroughly conversant with materials regarding the life of the subject, and an ample bibliography provides finger-posts directing the inquirer to more searching investigations. Most of the volumes of the series have a portrait frontispiece. In preparing his little volume Professor Loria discovered a discussion of Newton's laws in an eighteenth century manuscript which he described and reprinted in "Per la storia del newtonianismo in Italia," *Atti della Società italiana del Progresso delle Scienze*, Pisa, April, 1919, Rome, 1920.

NOTES.

Bulletin des Sciences Mathématiques for September, 1920, contains (pages 194–200) Picard's addresses at the opening and closing of the Strasbourg Mathematical Congress, September 22–30, 1920. They are also given in *Revue Scientifique*, November 13, 1920, pages 641–643.

In the *Quarterly Journal of Mathematics*, October, 1920, appears Sir Thomas Muir's seventh list of writings on determinants. It contains 264 titles, 32 of which are taken from this MONTHLY. Among the authors represented are 35 Americans.

The first 21 numbers (pages 1–254) of *Mathematical Notes, a Review of Elementary Mathematics and Science*, were published by the Edinburgh Mathematical Society, April, 1909–December, 1916. The next number of this publication did not appear until November, 1920, and then occupied pp. 51–64 of *Proceedings of the Edinburgh Mathematical Society*, volume 38.

Paul Appell's *Traité de Mécanique rationnelle* (Paris, Gauthier-Villars) has long been the leading treatise on the subject. The third edition ("entièrement refondue") of the third volume (*Equilibre et mouvement des milieux continus*, 674 pages, price 60 francs) and the first edition of a fourth volume (*Figures d'équilibre d'une masse fluide homogène en rotation sons l'attraction newtonienne*, 297 pages, price 30 francs) were published in November, 1920. Important additions to the third volume are the accounts of the work of Villat on movements of a fluid parallel to a fixed plane and of Bjerknes on "les fluides baroclines."

The concluding number of the first volume of *Archivio di Storia della Scienza* was published in August, 1920 [cf. *1920*, 474]. Aldo Miele continues his methodical bibliography (about 325 titles) of Italian works on the history of science, pages 397–420; there are also: (*a*) a brief review by A. Mieli of L. C. Karpinski, H. Y. Benedict, and J. W. Calhoun's *Unified Mathematics* (New York, 1918), 432; (*b*) "Elia Millosevich (1884–1919)" by G. Abetti, 446–447; (*c*) "H. G. Zeuthen (1839–1920)," with a list of his historical publications, by G. Loria, 447–451.

The following five mathematical periodicals have ceased publication (temporarily, at least): *Archiv der Mathematik und Physik* (last volume, 28, 1919); *Bibliotheca Mathematica* (last volume, 14, 1914–1915); *Journal de Mathématiques Pures et Appliquées* (Liouville, last volume, 84, 1919); *Nouvelles Annales de Mathématiques* (last volume, 79, 1920); *Zeitschrift für Mathematik und Physik* (last volume, 64, 1917). It has been announced that *Annales scientifique de l'Ecole Normale Supérieure* and *Bulletin de la Société Mathématique de France* will have to be discontinued unless many new subscribers are forthcoming at an early date.—Volume 81 of *Mathematische Annalen* (price, in Germany, 96

marks, 320 pages) was sent out by the new publisher, Springer, Berlin, in November, 1920. The editors plan to accept for publication articles on mathematics and all its applications, and not, as formerly, to limit publication to research papers in pure mathematics. Up to December 15, 1920, Teubner had published only the first number of volume 80.

The first number, June, 1920, of the second volume of *Bulletin de la Société Mathématique de Grèce* (*1920*, 314) opens with a list of the officers of the society and of its 123 members in 1919. This is followed by a portrait and brief sketch of "ΚΩΝΣΤΑΝΤΙΝΟΣ ΚΑΡΑΘΕΟΔΩΡΗ" (Constantine Carathéodory) recently appointed professor of mathematics at the University of Athens (*1920*, 337), but whom Vénizélos had expected would organize the new Greek university at Smyrna. The French articles in the number are: "Généralisation des formules de Combescure-Darboux" by N. Hatzidakis, 16–18; "Sur quelques propriétés des fonctions croissantes" by G. J. Rémoundos, 19–23; "Sur l'intégration de l'équation de Laplace entre deux sphères non concentriques" by D. Hondros, 24–28; "Sur la théorie de la flexion" by N. Sakellariou, 32–36.

The *Canadian Magazine* for July, 1920, contains an article by W. S. Wallace entitled "Some letters of Francis Maseres: 1766–1769." This interesting article furnishes new light on his life and on the period during which he was attorney general of Quebec.

Baron Maseres, historian, reformer and amateur mathematician, was born in London in 1731. He graduated from the University of Cambridge as senior wrangler in 1752 (according to Ball's *History of the Study of Mathematics at Cambridge*) and took up the practice of law. In 1773 he was appointed baron of the exchequer and he held this position until his death in 1824, "a length of tenure without parallel in the records of law." He was also for many years a senior judge of the sheriffs court in the city of London. "Homer he knew by heart, and Horace was at his fingers' ends. Lucian was his favorite next to Homer in ancient literature. . . . He spoke French fluently, but it was the language in idiom and expression which his ancestors had brought over to England." (*Dictionary of National Biography*).

Many of his numerous books, pamphlets, and reports, not referred to in '*D.N.B.*' are listed in H. J. Morgan, *Bibliotheca Canadensis*, 1867, and in P. Gagnon, *Bibliographie Canadienne*, Quebec, 1895. He wrote *Elements of plane trigonometry*, 1760, a meritorious treatise in two volumes on life assurance, 1783, and several works on algebra "which are valueless because he refused to allow the use of negative or impossible quantities." But his *Scriptores Logarithmici*, in six quarto volumes, 1791–1807, is a very useful work since it contains reprints of many of the older publications on logarithms.

Each of the three volumes of Wallis, *Opera Omnia*, 1699, in the Brown University library contains the autograph "F. Maseres. Aug. 27, 1774."

ARTICLES IN CURRENT PERIODICALS.

BULLETIN DES SCIENCES MATHÉMATIQUES, volume 55, August, 1920: Review by M. Fréchet of E. B. Wilson's *Aëronautics* (New York, 1920), 169–172 [Quotation: "Son livre a tous les mérites des Ouvrages de langue anglaise: simplicité, recours constant à l'intuition visuelle, théories constamment appuyées d'exemples numériques et concrets. Mais en même temps (pourrais-je sans exagération en reporter un peu le mérite au séjour en France par lequel il a terminé ses études, ou serait-ce tout simplement un don propre de M. Wilson) on trouve dans son livre une clarté d'exposition, un souci de l'ordre qui sont généralement des qualités reconnues aux Ouvrages français. Ces qualités, j'avais déjà eu le plaisir de les signaler dans son excellent Ouvrage *Advanced Calculus*, avec d'autant plus d'assurance que je me sentais sur un terrain plus familier"].

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 27, no. 2, November, 1920: "The twenty-seventh summer meeting of the American Mathematical Society" by A. Dresden, 49–65; "The Chicago colloquium" by W. A. Hurwitz, 65–71; "Note on velocity systems in curved space of n dimensions" by J. Lipka, 71–77; "Augustus De Morgan on divergent series" by F. Cajori, 77–81; "Russell's introduction to mathematical philosophy" by G. A. Pfeiffer, 81–90 [Review of Bertrand Russell's *Introduction to mathematical philosophy*. London, 1919]; Notes, 91–94; New publications, 94–96.

THE JOURNAL OF THE INDIAN MATHEMATICAL SOCIETY, volume 12, no. 1, February, 1920: "Multiplication of infinite integrals" by K. B. Madhava, 3–13; "Certain definite integrals and series" by C. Krishnamachary, 14–31; "Astronomical notes" by T. P. Bhaskara Sastri, 32–33; Questions and Solutions, 34–40—No. 2, April, 1920: "Functions of Legendre's type" by K. B. Madhava, 41–53; "Morley's theorem" by M. Bhimasena Rao, 54–58; "Note on the integration of $\int d\theta/(a - b\cos\theta)^n$" by K. B. Madhava, 59–60; "Note on the equation $a\sec\theta + b\operatorname{cosec}\theta = 1$" by M. T. Naraniengar, 60–61; Questions and Solutions, 61–80—No. 3, June: "The late Mr. S. Ramanujan, B.A., F.R.S." (portrait frontispiece) by P. V. Seshu Aiyar, 81–86 [Many details of his career and a complete list of his contributions to the *Journal of the Indian Mathematical Society*]; "In memoriam, S. Ramanujan" by R. Ramachandra Rao, 87–90 [Extracted from *Everyman's Review*, volume 5, nos. 7 and 8, 1920]; "S. Ramanujan, F.R.S." by G. H. Hardy, 90–91 [From *Nature*, June, 1920; compare this *Monthly, 1920*, 316, 338]; "Jean-Gaston Darboux, 1842–1917," by K. B. Madhara, 92–97; "On the set of points $\phi(n)/n$" by T. Vijayaraghavan, 98–99; "The generating planes of a quadric in five dimensions" by R. Vythynathaswamy, 100–101; Questions and Solutions, 101–119.

MATHEMATICAL GAZETTE, volume 10, October, 1920: "Imaginaries in geometry, and their interpretation in terms of real elements" by C. V. H. Rao, 129–133 [First paragraph: "The principal ideas are due to von Staudt and Lüroth. There is also a paper by Prof. Mathews, though it does not go far enough; and a recent treatise in English on the subject makes no reference at all to the question. It may be useful therefore to put down a brief account of the leading ideas of the two writers first mentioned, together with some simple developments."]; "Gleanings far and near," 133, 149; "The sound ranging problem" by G. Greenhill, 134; "Summation of harmonic progressions" by W. J. Ricketts, 135; "Some propositions relative to a tetrastigm" by J. H. Lawlor, 135–139; "The volume of a frustum of a sphere" by F. W. Russell, 139; "On the conic in polar coördinates," 140–141; "A method of finding the normal acceleration in circular motion," 142–143; "Brocard points for a quadrilateral" by R. W. Genese, 143–144; "Probability and athletic sports" by W. Hope-Jones, 144–145; "Some incidental writings of DeMorgan" (continued) 146–149; Reviews, 150–159 [pages 155–159: the first part of a review of A. Macfarlane's *Lectures on Ten British Physicists* (New York, 1919)].

MESSENGER OF MATHEMATICS, volume 50, no. 1, May, 1920: "4-tic & 3-bic residuacity-tables" by A. Cunningham and T. Gosset, 1–16.

NATURE, volume 106, October 28, 1920: "Elementary geometry" [review of C. Godfrey and A. W. Siddons's *Practical Geometry* and *Theoretical Geometry: a Sequel to Practical Geometry* (Cambridge, 1920)], 273–274—November 18: "The surveyor's art" [review of G. L. Hosmer's *Geodesy: including Astronomical Observations, Gravity Measurements, and Method of Least Squares* (New York and London, 1919)] by E. H. H., 369.

NOUVELLES ANNALES DE MATHÉMATIQUES, volume 79, September, 1920: "Surfaces de translation applicables l'une sur l'autre" (continued) by B. Gambier, 321–341; "Deux notes de

géométrie vectorielle" by R. Garnier, 341–347; "Un théorème général sur les complexes" by C. Servais, 347–355; "Démonstration géométrique du théorème de Liouville sur le groupe isogonal de transformations dans l'espace" by A. Lévêque, 356–358; "Chronique," 359–360—October: "Surfaces de translation applicables l'une sur l'autre" (conclusion) by B. Gambier, 361–372; "Sur une classe d'équations différentielles qui admettent des intégrales singulières" by E. Goursat, 372–395; "Sur des systèmes articulés" by R. Bricard, 395–400.

PACIFIC REVIEW, University of Washington, volume 1, no. 1, June, 1920: "And still it moves" by E. T. Bell, 83–92 [First two paragraphs: "It is related that Galileo on rising from his knees after having 'abjured, cursed and utterly detested' the heresy of the Earth's motion, muttered in his beard 'And still it moves.' The story is an epitome of the scientific temper. For in one respect the truths of science are like murder: they will out, no matter to whom they are distasteful, costly or fatal.

"'And still it moves.' Those four words sum up the history of science. They remind each generation of a perennial truth which its predecessors learned and forgot, scientific progress is inevitable and to be arrested by none. It is not mere pessimism to hold that each generation, profiting little or not at all by the bitterly won victories of those that have gone before, has had to master this fact anew; it is plain fidelity to the undisputed records of scientific history. In short it seems to be a law of human nature that the radically new shall be violently resented, provided only that it is also true. Let us trust that our own generation, tolerating rather than obstructing the onward rush of science, will be the first but not the last to break this grand law of human stupidity"].

PROCEEDINGS OF THE EDINBURGH MATHEMATICAL SOCIETY, volume 38, session 1919–20, November, 1920: "A geometrical proof of Professor Morley's extension of Feuerbach's Theorem" by H. W. Richmond, 2–5 [Morley's extension: "All curves of class three which (i) touch the six lines *OP, OQ, OR, QR, QP, PQ* joining four orthocentric points, *O, P, Q, R,* and (ii) pass through the circular points, also touch the common nine-points circle of the triangles *PQR, OQR, ORP, POQ*"]; "A proof of the binomial theorem, with some applications" by G. A. Gibson, 6–9; "The modified Bessel function $K_n(z)$" by T. M. MacRobert, 10–19; "Extension of Frenet's formulæ to a curve in flat space of n dimensions" by R. F. Muirhead, 20–23; "A version of Hagen's proof of the 'law of error'" by R. F. Muirhead, 24–26; "Note on the polynomials which satisfy the differential equation $x\frac{d^2y}{dx^2} + (\gamma - x)\frac{dy}{dx} - \alpha y = 0$" by N. McArthur, 27–33; "On direct and inverse interpolation by divided differences" by G. Smeal, 34–50; *Mathematical Notes, a review of elementary mathematics and science;* 51–64 [The last of the "notes" is "A link slide rule for the mechanical solution of quadratic equations" by G. D. C. Stokes, 61–64].

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES OF THE U. S. A., volume 6, no. 9, September, 1920: "On a differential equation occurring in Page's *Theory of Electromagnetism*" by H. Bateman, 528–529; "A new proof of a theorem due to Schoenflies" by J. R. Kline, 529–531; "On the structure of finite continuous groups with exceptional transformations" by S. D. Zeldin, 541–543.

PUNCH, London, volume 159, November 3, 1920: "Euclid in real life" by A. P. H., 346 [Illustrated by discussion of the problem: To find the center of a given circle, and the theorem: Any two sides of a triangle are together greater than the third side].

QUARTERLY JOURNAL OF PURE AND APPLIED MATHEMATICS, volume 49, no. 1, October, 1920: "An American tournament treated by the calculus of symmetric functions" by P. A. MacMahon, 1–36; "An approximation to $\frac{2}{\sqrt{\pi}}\int_0^a e^{-t^6}dt$" by H. B. C. Darling, 36–39; "On the partitions into unequal and into uneven parts" by P. A. MacMahon, 40–45; "On class number relations and certain remarkable sums relating to 5 and 7 squares" by E. T. Bell, 45–51; "A seventh list of writings on determinants" by T. Muir, 51–73; "The Stieltjes' integral and its generalisations" by S. Pollard, 73–96.

REVUE GÉNÉRALE DES SCIENCES, volume 31, September 15–30, 1920: "Le tricentenaire de l'abbé Picard" [1620–1684?] by E. Doublet, 561–564 [notable astronomer and professor at the Collège de France]. See page 123 of this issue of the MONTHLY.

SCHOOL SCIENCE AND MATHEMATICS, volume 20, no. 8, November, 1920: "The mathematics of elementary physics" by P. F. Hammond, 714–722; "A geometric recreation" by Isabel Harris, 731–733; "The graphical solution of spherical triangles" by M. O. Tripp, 734–742; Problems and solutions, 743–745—No. 9, December: "Value of the history of mathematical

ignorance" by G. A. Miller, 813–817; "Solutions of cubic equations by straight line graphs" by M. G. Schucker, 818–820; "The parallel development of mathematical ideas, numerically and geometrically" by L. C. Karpinski, 821–828; "Prose problems of algebra" by J. A. Nyberg, 829–835; "A study in determinants" by C. M. Himel and C. A. Stone, 835–837; Problems and solutions, 854–858.

SCIENCE, new series, volume 52, October 29, 1920: "Galileo's experiments from the tower of Pisa" by F. Cajori, 409; "Jonathan Edwards on multidimensional space and the mechanistic conception of life" by J. M. C., 409–410; "Note on Einstein's theory of gravitation and light" by E. Kasner, 413–414—November 12: "Predilection and sampling of human heights" by E. G. Boring, 464–466—November 26: "The American Mathematical Society" by F. N. Cole, 518–519 [report of the meeting held in New York on October 30, 1920]—December 10: "The History of Science Section and the progress of science" by F. E. Brasch, 559–562.

UNTERRICHTSBLÄTTER FÜR MATHEMATIK UND NATURWISSENSCHAFTEN, volume 26, nos. 5–6, August 20, 1920: "Ueber die geometrische Bedeutung von $\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}$" by E. Eckhardt, 57–61; "Verhandlungen über die Meraner Reform-Vorschläge," Mathematik, 61–65.

YALE ALUMNI WEEKLY, volume 30, December 3, 1920: "Teaching the freshmen—Mathematics in the freshman year" by W. R. Longley, 259–260 ["Three types of students elect mathematics in the Freshman year of Yale. Those of the first type choose mathematics for some reason best known to themselves but not because they expect to use it as a tool or because it is prerequisite for later work in the college course which they will probably take. Those of the second type need some mathematical training in Freshman year in order to follow successfully some of the later college courses for which one year of mathematics is prescribed. The students of the third type take Freshman mathematics as the first part of a more extensive course necessary for engineering study or specialized work in advanced mathematics, physics, or chemistry. . . . The topics covered are, necessarily, only those of an elementary and fundamental character, and the time is divided about equally between analytic geometry and calculus. As taught to Freshmen in Yale, and in most other colleges in this country, analytic geometry is no longer the exhaustive treatment of the multitudinous properties of conic sections that some graduates of twenty years ago may remember. The present course consists chiefly of methods for treating graphically problems of the kind arising in engineering, physics, chemistry, and other fields. Some practice is given in methods of calculation by numerical tables and the slide rule. In the part of the course devoted to calculus the aim is to present the fundamental ideas and concepts of the subject, rather than to develop any particular degree of skill in the manipulation of the machinery. . . . For those Freshmen whose entrance preparation in trigonometry and solid geometry is deficient, a course involving five exercises per week is provided. The instruction in analytic geometry and calculus in this course is identical with that in the shorter course and the extra time is used for trigonometry and solid geometry. The work is so arranged that any student deficient in only one of the entrance subjects mentioned is required to take the longer course for only one term. In the teaching of trigonometry and solid geometry the emphasis is laid upon the definitions and ideas necessary for later work and upon the use of formulas of a practical nature. The usual logical development of solid geometry is not followed; the obvious facts are merely stated and formal proofs are given only when necessary to establish mensuration formulas which can not be understood otherwise."]—January 14, 1921: "Death of Professor John E. Clark," 414 [Teacher of mathematics in Yale University 1872–1901].

ZEITSCHRIFT FÜR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT, volume 51, nos. 9–10, published September 14, 1920: "Kopfgeometrie" by B. Kerst, 217–223; "Die Apollonische Berührungsaufgabe" by E. Salkowski, 224–230; "Das Apollonische Berührungsproblem in stereometrischer Behandlung" by A. Lanner, 231–239; "Atom- und Molekularwärmen fester Körper" by L. Müller, 239–246; "Kleine Mitteilungen," 246–248; "Bücherbesprechungen," 254.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them.* If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2883. Proposed by ROBERT C. COLWELL, Geneva College, Beaver Falls, Pennsylvania.

Evaluate (*a*) $\int_0^\pi \cos^2\left(\frac{n\pi\cos\theta}{2}\right)d\theta$ and (*b*) $\int_0^\pi \frac{\cos^m\theta d\theta}{\theta^n}$.

2884. Proposed by E. H. MOORE, University of Chicago.

Consider an $m \times n$ array A of numbers a_{st} and an $n \times m$ array B of numbers b_{ts}. Show that the system of mn equations:

$$\sum_{tu} a_{st}b_{tu}a_{uv} = 0 \qquad (sv),$$

implies the equation:

$$\sum_{ts} a_{st}b_{ts} = 0.$$

The suffixes s, u have the range 1, 2, $\cdots$, m and the suffixes t, v have the range 1, 2, $\cdots$, n.

2885.

If A, B, C, X, Y are given collinear points, construct Z so that $\{ABCX\}+\{ABCY\}=\{ABCZ\}$, where $\{ABCX\}$ denotes the cross-ratio of the points A, B, C, X. [From *the Math. Tripos Exam.*, Cambridge, Eng., 1905.]

2886. Proposed by S. A. COREY, Des Moines, Iowa.

If, in considering the purchase of bonds in the open market, x = interest yield to investor; i = interest rate named in bond; t = time in years to maturity; and q = quoted market price, the relation between these quantities is expressed by the equation, $xq - (x - i)(1 + x)^{-t} - i = 0$. Find value of x.

Also show that, if $a = i/q + (1 - q)/qt$, we may, in practice, safely assume that very nearly,

$$x = \frac{at(a - i) + i[(1 + a)^{t+1} - (1 + a)]}{q(1 + a)^{t+1} + t(a - i) - (1 + a)}.$$

2887. Proposed by the late L. G. WELD.

A carpenter's square moves with its outer edges in contact with two round pegs of given equal diameters. Define the locus of the "heel" of the square.

2888. Proposed by J. A. BULLARD, U. S. Naval Academy.

Discuss the surface, $x^{1/2} + y^{1/2} + z^{1/2} = a^{1/2}$, by means of plane sections. (Cf. problem 2846, *1920*, 326.)

2889. Proposed by W. D. LAMBERT, U. S. Coast and Geodetic Survey.

Evaluate, $\int_c^1 \frac{\cos^{-1} x dx}{\sqrt{x^2 - c^2}}$, where $0 < c < 1$.

PROBLEMS—NOTES.

7. The following problem is discussed in *Revista Matemática Hispano-Americana*, 1920, pages 190–191, 226–228: "What is the maximum number of spheres a decimeter in diameter that can be placed in a box in the form of a cube a meter along any interior edge?" It is found that this number is 1254. Professor R. P. Baker's problem, 501 (geometry), twice proposed in this MONTHLY [*1916*, 341; *1919*, 414], but not yet solved, may be recalled in this connection: "Find the minimum amount of lumber one inch thick required to pack a gross of spheres three inches in diameter in a rectangular box." ARC.

8. Perfect Numbers. A number which equals the sum of its divisors other than itself is called perfect. Euclid [about 300 B.C.] proved that if

$$p = 1 + 2 + 2^2 + \cdots + 2^{n-1}$$

is a prime, $2^{n-1} \cdot p [= 2^{n-1}(2^n - 1)]$ is a perfect number. So far as known, the first six perfect numbers are: 6, 28, 496, 8128, 33550336, and 8589869056. A posthumous paper of L. Euler [1707–1783] contains the proof that every even perfect number is of Euclid's type.[1] He proved also that every odd perfect number must be of the form $r^{4\lambda+1}P^2$, where r is a prime of the form $4n + 1$. No odd perfect number is known. It has been proved that there is no odd perfect number less than two million, or with less than five distinct prime factors.[2]

Mersenne stated, in effect, in the preface of his *Cogitata Physico-Mathematica*, Paris, 1644, that the first eleven perfect numbers are $2^{n-1}(2^n - 1)$ for $n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 167, 257$. Since p can be a prime only when n is a prime the problem of verifying the statement, so far as even perfect numbers are concerned, is equivalent to that of determining when p is prime for the 56 prime values of n less than 258. This has been effected for 45 of the prime values and the untested cases are for $n = 137, 139, 149, 157, 167, 193, 199, 227, 229, 241$, and 257. Mersenne's statement has been shown to be incorrect in at least five cases, namely when $n = 61, 89, 107, 127$, and 67. In the first four of these cases p has been shown to be prime. F. N. Cole found the two prime factors of p when $n = 67$, *Bulletin of the American Mathematical Society*, volume 10, page 137, 1903. America's further contribution to the discussion of perfect numbers was made through R. E. Powers who verified, in 1911, that p is prime when $n = 89$; proved, in 1914, that for $n = 107$, p is prime; and, in 1916, showed that for $n = 103$ or 109, p is composite.[3] According to *Sphinx-Oedipe*, July, 1914, and February, 1920, E

[1] The neatest proof of this was given in six lines by L. E. Dickson, in this MONTHLY, *1911*, 109.

[2] In a paper on perfect numbers by Benjamin Peirce, "Mathematical Instructor in Harvard University," *The New York Mathematical Diary*, no. 13, vol. 2, pp. 267–277, 1832, it is shown that there can be no odd perfect number "included in the forms a^r, $a^r b^s$, $a^r b^s c^t$, where a, b, and c are prime numbers and greater than unity." L. E. Dickson, in the work referred to on the opposite page, does not mention this paper. He does record: in 1844 "V. A. Lebesgue stated that he had a proof that there is no odd perfect number with fewer than four distinct prime factors." We now see that an American mathematician gave a *proof* of this theorem twelve years earlier.

[3] In his inaugural lecture before the University of Oxford (*Some Famous Problems of the Theory of Numbers and in particular Waring's Problem*, Oxford, 1920) G. H. Hardy remarked: "We have seen this ["tendency to exaggerate the profundity implied by the enunciation of a

Fauquembergue showed p prime for $n = 107$ and 127. Hence to sum up, p is known to be prime for $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107$, and 127; and therefore 12 perfect numbers are known as such. A. Cunningham announced (*Report, British Association for the Advancement of Science*, 1895, p. 614) 12 other values of n ranging in value from $n = 317$ to $n = 132{,}019$, for each of which p is composite. Niewiadomski found a similar result for $n = 761$.

"It must always," wrote Sylvester in 1888, "stand to the credit of the Greek geometers that they succeeded in discovering a class of perfect numbers which in all probability are the only numbers which are perfect." (*Collected Mathematical Papers*, volume 4, 1912, page 589.) Sylvester referred also to the question of the non-existence of odd perfect numbers as "a *problem of the ages* comparable in difficulty to that which previously to the labors of Hermite and Lindemann . . . environed the subject of the quadrature of the circle" (*ibid.*, page 626).

The authoritative reference work which should be consulted by those desiring to learn what has been done in connection with perfect numbers is L. E. Dickson, *History of the Theory of Numbers*, volume 1, 1919. W. R. Ball's statements in the chapter on "Mersenne's numbers," and elsewhere in his *Mathematical Recreations and Essays* (fifth edition, 1911), should be carefully checked with Dickson's work. ARC.

9. In *Norsk Matematisk Tidsskrift*, the official organ of the Norwegian Mathematical Society, the following question was proposed in 1919, volume 1, page 80: "Usually x^y is greater or less than y^x; but for special values of x and y they are equal to one another, e.g., $2^4 = 4^2$. What are other values of x and y for which x^y and y^x are equal?" A reply by T. Nagel appears in the third number of the *Tidsskrift* for 1920, pages 93–94.

Mr. Philip Franklin, of Princeton University, has kindly informed me that in seeking the equations of all curves having the property that their evolutes are equal curves [compare *1920*, 303–306] he was led to a solution involving the roots of the transcendental equation $x = e^{ax}$. If x and y are two roots of this equation[1]

$$x^y = y^x = e^{axy}.$$

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theorem"] even in the case of Fermat, a mathematician of a class to which Waring had not the slightest pretensions to belong, whose notorious assertion concerning 'Mersenne's numbers' has been exploded, after the lapse of over 250 years, by the calculations of the American computer Mr. Powers." Such a statement from such an authority, and on such an occasion, was extraordinary. So far as known Fermat made no erroneous statement whatever in connection with Mersenne's numbers. It is learned that the fanciful conjectures of Ball were Professor Hardy's only authority. *Mersenne's* inaccuracies and Mr. Powers's correction of *them* have been noted above.

[1] K. Schwering showed in 1878 (*Zeitschrift für Mathematik und Physik*, vol. 23, pp. 339–343) that solutions of the equation $x^y = y^x$ were obtained by taking any two solutions of the equation $a^x = x$. He showed also that for this latter equation there are an infinite number of solutions given by complex values for x. In 1896–1897, E. M. Lémeray found a number of results in connection with this equation (*Nouvelles Annales de Mathématiques*, vol. 16, pp. 548–556; and vol. 17, pp. 54–61). Among these were the following: when a is between 0 and 1, the equation $a^x = x$ has a real root between 0 and 1; when a is between 1 and $e^{1/e}$ there is a real root betwee 1 and e and a second real root between e and ∞ [$e^{1/e} = 1.444667\cdots$; for $a = \sqrt[3]{3} = 1.442249\cdots$ we have as roots of $(\sqrt[3]{3})^x = x$, $x_1 = 3$ and $x_2 = 2.478055\cdots$ (E. Heis, *Sammlung von Beispielen*

He showed in this MONTHLY, *1916*, 233–237, that the polar equation of $x^y = y^x$ is $r = \sec\theta \cdot (\tan\theta)^{1/(\tan\theta - 1)}$ and that its parametric equations are $x = m^{1/(m-1)}$ and $y = m^{m/(m-1)}$.

But the discussion of the curve goes back much farther; indeed these parametric equations were given by Euler in his *Introductio in analysin infinitorum*, tome 2, Lausanne, 1748, p. 294 (French edition, 1797, p. 297). Setting $m - 1 = 1/u$ he gave also the form $x = \left(1 + \frac{1}{u}\right)^u$ and $y = \left(1 + \frac{1}{u}\right)^{u+1}$ and continued: "Thus the curve has, in addition to the line EAF, the branch RS which converges towards the lines AG and AH as asymptotes[1] and of which AF will be a diameter. Further the curve will cut the line AF at the point C where $AB = BC = e$, e denoting the number whose logarithm is unity. The equation furnishes also an infinity of separate points which with the line EF and the curve RCS exhaust those defined by it. There is, then, an infinity of numbers x and y which taken two and two satisfy the equation $x^y = y^x$; such are the following numbers, among these which are rational: $x = 2$, $y = 4$; $x = 9/4$, $y = 27/8$; $x = 64/27$, $y = 256/81$; $x = 625/256$, $y = 3125/1024$; etc." T. Nagel proved that: $x = 2$, $y = 4$, is the only pair of positive *integers* $(y > x)$ satisfying the equation. C. Herbst showed[2] in 1909 that, if $|y| > x$, $x = 2$, $y = 4$ and $x = -2$, $y = -4$ are the only integers satisfying the relation. These negative values were, apparently, overlooked by Daniel Bernoulli when writing to Goldbach[3]

und Aufgeben aus ··· Arithmetik und Algebra. 75. Aufl., Köln, 1888, p. 371).] For $a = e^{1/e}$ there is a double root whose value is e. When $a = e$, Cauchy found (*Leçons sur le calcul différentiel*, Paris, 1829, leçon 14) $x = 0.3181317 \pm 1.3372357\sqrt{-1}$. By another method Stern got (Crelle's *Journal*, vol. 22, 1841, p. 59) $x = 0.318133 \pm 1.337238\sqrt{-1}$. The expression converging to the value of a root for a given a is $a^{a^{a^{\cdot^{\cdot^{\cdot}}}}}$. This expression was studied by G. Eisenstein and F. Woepcke (Crelle's *Journal*, vol. 28, 1844, pp. 49–53; vol. 42, 1851, pp. 83–90). Certain errors of Eisenstein were corrected by P. L. Seidel who found, in effect, the Léméray results noted above (*Abhandlungen der kgl. Baierischen Akademie der Wissenschaften*, zweite classe, vol. 11, 1870). The rôle that the equation $\omega^\xi = \xi$ plays in Cantor's theory of transfinite numbers will be recalled; see, for example, G. Cantor, *Mathematische Annalen*, vol. 49, 1897, pp. 242–246 (also English translation by P. E. B. Jourdain, Open Court Publ. Co., 1915, pp. 195–201).

Reference may be given also to: Hessel, (1) "Ueber die Bedingung unter welcher $a^x > x$ ist," (2) "Ueber das merkwürdige Beispiel einer zum Theil punctirt gebildeten Curve das der Gleichung entspricht: $y = \sqrt[x]{x}$," *Archiv der Mathematik und Physik*, vol. 14, 1850, pp. 93–96, 169–187; H. Scheffler, "Ueber die durch die Gleichung $y = \sqrt[x]{x}$ dargestellten Curven," *Archiv d. Math. u. Physik*, vol. 16, 1851, pp. 133–137; L. Oettinger. "Ueber den grossten Werth von $\sqrt[x]{x}$ und einige damit zusammenhängende Sätze," *Archiv d. Math. u. Physik*, vol. 42, 1864, pp. 106–113; and to L. Moreau: (1) *Analyse ou nombre de solutions et fixations des racines rémarquables de l'équation* $a^x = x$. (Brussels, 1900, 8vo, 16 pages); (2) "Variation du rapport a^z/z d'un nombre à son logarithme," *Journal de Mathématiques Spéciales*, vol. 25, pp. 170–173, 1900. Nomographic discussions of $a^x = bx$ and $a^x = x^b$ are given in S. Brodetsky, *A First Course in Nomography*, London, 1920, pp. 130–133.

[1] Euler's error here in stating that the coordinate axes are asymptotes (instead of $x = 1$ and $y = 1$) does not appear to have been previously remarked. The four asymptotes ($x = \pm 1$, $y = \pm 1$) were first correctly given in 1913 by A. M. Nesbitt, *Mathematical Questions and Solutions from 'The Educational Times,'* n.s. vol. 23, pp. 77–78, where the curve is plotted and discussed. Somewhat fuller considerations of a similar character were given by E. J. Moulton in this MONTHLY, *1916*, 233–237.

[2] *Unterrichtsblätter für Mathematik und Naturwissenschaften*, Jahrgang 15, pp. 62–63.

[3] *Correspondance Mathématique et Physique* (Fuss), vol. 2, 1843, p. 262.

June 29, 1728: "Je finirai par un problème qui m'a paru fort curieux et que j'ai résolu. Le voici: Trouver deux nombres inégaux x et y tels que $x^y = y^x$. Il n'y a qu'un cas ou ces nombres soient entiers, savoir $x = 2$ et $y = 4$ (car $2^4 = 4^2$), mais on peut donner une infinité de nombres rompus qui satisfont au problème. Il y a aussi d'autres espèces de quantités dont je ne dirai rien." [1] In reply to this on January 31, 1729, Goldbach wrote as follows[2] (*l.c.* pp. 280–281): "Je ne trouve pas la moindre difficulté à faire voir que, dans l'équation $x^y = y^x$, les nombres x et y ne peuvent être entiers à moins que l'un ne soit $= 2$, et l'autre $= 4$, et que, pour les nombres rompus, on peut donner une infinité de solutions. Voici comment je m'y prends: Je fais $y = ax$, donc $x^{ax} = a^x x^x$ et enfin $x = a^{1/(a-1)}$. Or, il est visible que x ne peut être un nombre entier que dans la supposition de $a = 2$; car si a est un nombre entier plus grand que 2, on voit d'abord que x devient irrationel; d'un autre côté, a étant un nombre rompu, toutes ses puissances seront autant de nombres rompus, et par conséquent x ne peut être un nombre entier; mais pour exprimer la valeur de x par des nombres rompus, il n'y a qu'à faire

$$x = f^{g/(f-g)} : g^{g/(f-g)}$$

où f et g soient des nombres entiers." [3]

ARC.

PROBLEMS—SOLUTIONS

2791 [1919, 414]. A cup of wine is suspended over a cup of equal capacity full of water; through a small hole in the bottom, the wine drips into the water, and the mixture drips out at

[1] Concerning this passage Cantor remarks (*Vorlesungen über Geschichte der Mathematik*, vol. 3_2, 1901, p. 610): "Man wird nach diesem Schlussworte wohl oder übel annehmen müssen, dass Bernoulli an complexe Auflösungen dachte." One must agree with Eneström (*Bibliotheca Mathematica*, vol. 13 (3), p. 270) that this surmise is "höchst unwahrscheinlich." The natural interpretation is that Bernoulli referred to the infinite number of irrational values of x and y which are obtained from Euler's equations given above, when u is not integral.

[2] The argument of Herbst is very similar.

[3] Other discussions of the relation $x^y = y^x$ are: T. Wittstein, *Archiv der Mathematik und Physik*, vol. 6, pp. 154–162, 1845; I. L. A. Lecointe, *Nouvelles Annales de Mathématiques*, vol. 11, pp. 187–189, 1852; M. Cantor, *Zeitschrift für mathematische und naturwissenschaftlichen Unterricht*, vol. 9, pp. 163–164, 1878; L. F. Marrecas Ferreira, *Jornal de Sciencias Mathematicas e Astronomicas*, vol. 2, pp. 165–166, 1880; M. Luxemberg, *Archiv der Mathematik und Physik*, vol. 66, pp. 332–334, 1881; D. Besso, U. Danielli, and L. Carline, *Periodico di Matematica per l'Insegnamento Secondario*, vol. 5, pp. 12–15, 115–117, 117–119, 1890; A. Flechsenhaar, and R. Schimmack, *Unterrichtsblätter für Mathematik und Naturwissenschaften*, vol. 17, pp. 70–73, 1911 and vol. 18, pp. 34–35, 1912; A. Tanturri, *Periodico di Matematica . . .*, vol. 30, pp. 186–187, 1915.

In Nouvelles Annales de Mathématiques, 1876, Moret-Blanc proved (pp. 44–46) that the only positive integral solutions of the equation $x^y = y^x + 1$, are $y = 0$, x arbitrary; $y = 1$, $x = 2$; $y = 2$, $x = 3$; also Meyl proved (pp. 545–547) that the only positive integral solutions of the equation $(x+1)^y = x^{y+1} + 1$, are $x = 0$, y arbitrary; $x = 1$, $y = 1$; $x = 2$, $y = 2$. Landau showed (*L'Intermédiaire des Mathématiciens*, 1901, pp. 151–152) that the solutions found by Moret-Blanc for the equation $x^y = y^x + 1$ are the only positive rational solutions. This equation may be obtained from the simultaneous equations $x^y = 3$, $y^x = 2$ for which E. Heis gave the approximate solution $x = 2.23925113$, $y = 1.36280365$ (*Sammlung von Beispielen und Aufgaben aus . . . Arithmetik und Algebra*. 75. Aufl., Köln, 1888, p. 372; *Ausführliche Auflösuug der in Dr. Ed. Heis' Sammlung von Biespielen enthaltenen Aufgaben*, Dritter Theil, Bonn, 1880, pp. 386–388).

In *Messenger of Mathematics*, A. Cunningham discussed the factorization of $x^y - y^x$, x and $y > 1$, and x prime to y (April, 1916, vol. 45, pp. 185–192), and of $2^x - x$, x positive (May, 1917, vol 47, pp. 1–38).

the same rate. When the wine cup is empty, what part of the contents of the lower cup is water? [Proposed by Charles Gilpin, Jr., Philadelphia, as Problem 287 in *The Mathematical Visitor*, January, 1881, volume 1, page 193. No solution was published in the *Visitor*.]

I. Solution by C. A. Noble, University of California.

Professor Hurwitz has called attention to the fact that this problem is, except for phraseology, a special case of the note by Professor Noble in this Monthly, *1919*, 191–194. In the formula for s on page 193, if we put $s_0 = 0$, $s_1 = 1$, and $v = v_0$, we have for s (the proportion of wine) $1 - e^{-1}$, so that also the proportion of water is e^{-1}.

II. Solution by H. L. Olson, University of Michigan.

Under the hypotheses the liquid will drip from the lower cup (and hence also from the upper cup) at a constant rate, $a\sqrt{2gh}$, where a is the area of the hole, h the height of the liquid in the lower cup, and g the acceleration of a freely falling body under gravity. Let q be the capacity of each cup, and x the amount of water in the lower cup. Then

$$\frac{dx}{dt} = -\frac{a\sqrt{2gh}}{q}\,x.$$

Making use of the fact that when $t = 0$, $x = q$, we have

$$x = qe^{-\frac{at\sqrt{2gh}}{q}}.$$

When the upper cup is empty, $t = \dfrac{q}{a\sqrt{2gh}}$ and $x/q = e^{-1}$,

the proportion of water in the lower cup at that time.

Note.—The problem does not say that the liquid will drip from the cups at a constant rate, but only at the same rate. If it drips from the first cup under the influence of gravity the rate will not be constant. But the rate is not essential; we can assume any rate, e.g., the constant rate $a\sqrt{2gh}$, whatever a and h may mean, and get the correct result.—Editors.

III. Solution by R. E. Gaines, Richmond College.

A generalization of this problem leads to a remarkable result.

Let there be a series of cups of equal capacity full of water and arranged one below another. Pour into the first cup an equal quantity of wine at a constant rate and let the overflow in each cup go into the cup just below it. Assuming that complete mixture takes place instantaneously, find the amount of wine in each cup at any time t, and at the end of the process, at time T.

Let q denote the capacity of each cup, so that the rate of flow is q/T.

If x_K be the amount of wine in the Kth cup and x_{K-1} that in the $(K-1)$th cup, then the rate of flow of wine into the Kth cup is x_{K-1}/T and out of it is x_K/T. Whence,

$$\frac{dx_K}{dt} = \frac{x_{K-1}}{T} - \frac{x_K}{T}.$$

Any equation in the series can be solved if the one before it has been solved. So we begin at the first cup and have

$$\frac{dx_1}{dt} = \frac{q}{T} - \frac{x_1}{T}. \quad \text{Hence, } x_1 = q\left(1 - \frac{1}{e^{t/T}}\right);$$

$$\frac{dx_2}{dt} = \frac{q\left(1 - \frac{1}{e^{t/T}}\right)}{T} - \frac{x_2}{T}. \quad \text{Hence, } x_2 = q\left(1 - \frac{1 + \frac{1}{1!}\frac{t}{T}}{e^{t/T}}\right).$$

Similarly,

$$x_3 = q\left(1 - \frac{1 + \frac{1}{1!}\frac{t}{T} + \frac{1}{2!}\left(\frac{t}{T}\right)^2}{e^{t/T}}\right);$$

and so on.

For the *final* amounts in the successive cups, we have

$$X_1 = q\left(1 - \frac{1}{e}\right), \quad X_2 = q\left(1 - \frac{1 + \frac{1}{1!}}{e}\right), \quad X_3 = q\left(1 - \frac{1 + \frac{1}{1!} + \frac{1}{2!}}{e}\right), \ \cdots.$$

In general, we have

$$X_K = q\left(1 - \frac{e_K}{e}\right)$$

where e_K is the sum of the first K terms of

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \text{etc.}$$

As $K \doteq \infty$, $X_K = 0$.

Note.—Here also the rate of flow is not essential. If we take as the independent variable the amount of wine x which has been poured into the first cup, then the differential equations are

$$\frac{dx_1}{dx} + \frac{x_1}{q} = 1, \qquad \frac{dx_k}{dx} + \frac{x_k}{q} = \frac{x_{k-1}}{q}$$

and

$$x_k = q\left[1 - \left(1 + \frac{x}{q} + \frac{1}{2!}\left(\frac{x}{q}\right)^2 + \cdots + \frac{1}{(k-1)!}\left(\frac{x}{q}\right)^{k-1}\right)e^{-x/q}\right].$$

The final result is obtained by setting $x = q$.—EDITORS.

Also solved by W. D. CAIRNS, ALEXANDER KNISELY, L. C. MATHEWSON, and ARTHUR PELLETIER.

2792 [1919, 414]. Proposed by B. J. BROWN, Kansas City.

Solve the differential equation,

$$x^2(1 - x)\frac{d^2y}{dx^2} + 2x(2 - x)\frac{dy}{dx} + 2(1 + x)y = x^2.$$

SOLUTION BY C. P. SOUSLEY, Pennsylvania State College.

This equation is exact and the first integral is,

$$x^2(1 - x)\frac{dy}{dx} + x(x + 2)y = \frac{x^3 + C}{3},$$

or

$$\frac{dy}{dx} + \frac{x + 2}{x(1 - x)}y = \frac{x^3 + C}{3x^2(1 - x)}.$$

Multiplying through by the integrating factor, $x^2/(1 - x)^3$, we have

$$\frac{x^2}{(1 - x)^3}\frac{dy}{dx} + \frac{x(x + 2)}{(1 - x)^4}y = \frac{x^3 + C}{3(1 - x)^4},$$

and on integrating, we have

$$\frac{x^2}{(1-x)^3}y = \frac{C+1}{9(1-x)^3} - \frac{1}{2(1-x)^2} + \frac{1}{(1-x)} + \log K.\sqrt[3]{(1-x)}.$$

Solved similarly by C. A. ISAACS, GERTRUDE MCCAIN, and H. L. OLSON.

2793 [1919, 458]. Proposed by J. L. RILEY, Stephenville, Texas.

If a, b, and c, are complex, and α, β, and γ, real constants, the point

$$x = \frac{at^2 + 2bt + c}{\alpha t^2 + 2\beta t + \gamma}$$

traces a conic or a straight line when t takes all real values.

DISCUSSION BY A. F. FRUMVELLER, Marquette University.

Since x is a complex number, let us put $x = u + iv$, $a = a_0 + a_1 i$, $b = b_0 + b_1 i$, $c = c_0 + c_1 i$, and clear of fractions. Separating the real and imaginary parts of this equation, we obtain the simultaneous set

$$(1) \qquad \begin{cases} t^2(\alpha u - a_0) + 2t(\beta u - b_0) + (\gamma u - c_0) = 0, \\ t^2(\alpha v - a_1) + 2t(\beta v - b_1) + (\gamma v - c_1) = 0. \end{cases}$$

The eliminant is $|\, p_0 q_1 \,| \cdot |\, p_1 q_2 \,| - |\, p_0 q_2 \,|^2 = 0$ (L. E. Dickson, *Elementary Theory of Equations*, New York, 1914, p. 155), where

$$|\, p_0 q_1 \,| = 2 \begin{vmatrix} \alpha u - a_0 & \beta u - b_0 \\ \alpha v - a_1 & \beta v - b_1 \end{vmatrix}$$

$= 2\,[(\alpha b_0 - a_0\beta)v + (a_1\beta - \alpha b_1)u + (a_0 b_1 - a_1 b_0)]$ with similar expressions for the other two determinants.

The eliminant is, therefore, a quadratic in (u, v), *i.e.*, a conic, which under suitable conditions degenerates into straight lines. This conic in the plane uov (the plane of the complex number x) is in reality the projection of the actual path of the moving point in space as it spirals its way around the axis of t or a parallel line standing out at right angles to the lines $\overline{ou}$, $\overline{ov}$, in the x-plane.

Cf. an article on "The graph of $f(x)$ for complex numbers" (this MONTHLY, *1917*, 409), where many analogous examples are worked out and graphed in this rather unusual system of coördinates.

Also solved by ARTHUR PELLETIER.

2796 [1919, 458]. Proposed by N. P. PANDYA, Amreli, India.

Construct a triangle ABC having its centroid on a given ellipse, AB being a fixed diameter of the ellipse and C lying on one of the directrices.

SOLUTION BY GRACE M. BAREIS, Ohio State University.

Let O be the center of the given ellipse. Construct OM perpendicular to a directrix and meeting it at M. Determine P on OM so that $OP = \frac{1}{3}OM$. Through P draw a line parallel to the directrix and cutting the ellipse in N_1 and N_2. Draw ON_1 and ON_2 meeting the directrix in C_1 and C_2, respectively. Then ABC_1 or ABC_2 is a solution. It is to be noted that the points C_1 and C_2 are fixed points whatever diameter AB may have been chosen. The problem has four solutions, two corresponding to each directrix, if $e > \frac{1}{3}$; two solutions, one corresponding to each directrix, if $e = \frac{1}{3}$; no real solution when $e < \frac{1}{3}$.

Also solved by E. J. OGLESBY, H. L. OLSON, and ARTHUR PELLETIER.

2797 [1919, 458]. Proposed by E. J. OGLESBY, New York University.

Solve for x and y, the simultaneous equations,

$$x^3 + y^3 = 35 \quad \text{and} \quad x^2 + y^2 = 13.$$

SOLUTION BY H. N. CARLETON, West Newbury, Mass.

Let $x = u + v$ and $y = u - v$. Substituting these values of x and y in the original equations, we have after reducing

$$2u^3 + 6uv^2 = 35 \cdots \quad (1)$$

and

$$2u^2 + 2v^2 = 13 \cdots . \quad (2)$$

Equating the values of v^2 from (1) and (2) we have

$$4u^3 - 39u + 35 = 0, \quad \text{or} \quad (u - \tfrac{5}{2})(4u^2 + 10u - 14) = 0.$$

Whence, $u = 5/2$, 1, or $-7/2$. Hence, corresponding values of v are $\frac{1}{2}$, $\frac{1}{2}\sqrt{22}$, and $\frac{1}{2}\sqrt{-23}$, respectively. Hence, $x = u + v = 3$, $1 + \frac{1}{2}\sqrt{22}$, or $-7/2 + \frac{1}{2}\sqrt{-23}$ and the corresponding values of y are $u - v = 2$, $1 - \frac{1}{2}\sqrt{22}$, and $-\frac{7}{2} - \frac{1}{2}\sqrt{-23}$, respectively. By virtue of the symmetry of the equations, the corresponding values of x and y may be interchanged, thus making a total of six solutions.

Also solved by H. L. AGARD, NORMAN ANNING, GRACE M. BAREIS, H. C. BRADLEY, S. M. BERG, JOHN BIGGERSTAFF, G. A. BINGLEY, W. E. CLELAND, H. J. ETTLINGER, C. H. GROVE, OLIVE C. HAZLETT, ALEX. KNISELY, C. N. MILLS, J. Q. MCNATT, A. R. NAUER, H. L. OLSON, R. H. REECE, J. L. RILEY, EMETERIO ROA, M. M. SMITH, H. S. UHLER, W. W. WARNER, E. E. WHITFORD, and C. N. WUNDER.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will cooperate in contributing to the genera interest of this department by sending items to the Editor-in-Chief.

Professor J. K. LAMOND, of Pennsylvania College, Gettysburg, has resigned to accept a position in the engineering department of the Bell Telephone Company at Philadelphia.

Dr. C. L. E. WOLFE, of the Junior College at Santa Rosa, has been appointed instructor in mathematics at the California Institute of Technology, Pasadena.

On page 187 of *Scientific Monthly*, February, 1921, there is a full-page portrait of Professor E. H. MOORE, of the University of Chicago, president of the American Association for the Advancement of Science.

At the close of the present academic year Dr. P. H. HANUS, professor of the history and art of teaching at Harvard University since 1901, will retire from active service. He was born in Silesia and came to the United States when he was four years old. He taught mathematics and science in a high school of Denver, Col. 1878–79, was professor of mathematics in the University of Colorado 1879–86 and was principal of another Denver high school 1886–90. His connection with Harvard dates from 1891. His first book, *An Elementary Treatise on the Theory of Determinants*, was published in 1886. There have been several reprints of his *Geometry in the Grammar School; an essay together with illustrative class exercises, and an outline of the work for the last three years of the grammar school* (4 + 52 pages, 1893).

JOHN EMORY CLARK, professor of mathematics, emeritus, in the Sheffield Scientific School, Yale University, died at Hartford, Conn., January 3, 1921. He was born in Northampton, N. Y., August 8, 1832. He graduated A.B. from the University of Michigan where he was an assistant professor of mathematics 1857–1859. He was professor of mathematics and astronomy at Antioch College from 1866 to 1872 when he became instructor in mathematics at Yale, and after a year was appointed to the chair of mathematics, which appointment he held until he was made professor emeritus in 1901. "Professor Clark will be remembered by his former students chiefly for his kindliness. Having himself an abstract and mathematical mind, he yet could endeavor to make his subject interesting to the members of his courses. He was perhaps not a great mathematician, but he was a great teacher, with patience and kindness beyond most, and his students held him in esteem and affection." (*Yale Alumni Weekly*, January 14, 1921.)

Doctor ALEXANDER PELL died at Bryn Mawr, Pa., January 26, 1921. He was born in Moscow, Russia, September 25, 1857. Entering the Artillery School at St. Petersburg in 1873, he graduated in 1878 as first lieutenant in the Russian artillery. He left the service in 1879 with the rank of captain and spent two years at the Institute of Civil Engineers in St. Petersburg. Forced by failure of revolutionary activities to leave his native country, he came to America in the early eighties. He was in a printing office of a Canadian village, and in a chemical factory of St. Louis, Mo., for sometime. Finally he entered The Johns Hopkins University as a graduate student in mathematics in October, 1895, was a fellow 1896–97, and received the degree of doctor of philosophy in June, 1897. His thesis entitled "On the focal surfaces of the congruences of tangents to a given surface" was published in *American Journal of Mathematics*, 1898. Dr. Pell's other mathematical papers were published in *Transactions of the American Mathematical Society*, 1900, in *Annals of Mathematics*, 1900, 1918, and in *Bulletin of the American Mathematical Society*, 1901. He was professor of mathematics and astronomy at the University of South Dakota, 1897–1908 (dean of the engineering school, 1906–1908), and assistant and associate professor of mathematics at the Armour Institute from 1908 to 1913. During the second semester of the year 1910–1911 he was incapacitated by a serious illness and Mrs. Pell acted as substitute. In one semester of 1915–1916 he was instructor in mathematics at Northwestern University. He married, for the second time, at Göttingen in 1907, Miss Anna Johnson, now associate professor of mathematics at Bryn Mawr College. A sketch of Dr. Pell's life by Professor Charlotte A. Scott was published in *The College News*, Bryn Mawr, February 9, 1921. This contains personal details not indicated above.

JOHANN MARTIN KRAUSE, ordinary professor in the Technische Hochschule, Dresden, since 1888, died March 2, 1920. He was born June 29, 1851. His article, "Ueber Systeme von Differentialgleichungen denen vierfach periodische Functionen Genüge leisten," in the first volume of *Transactions of the American Mathematical Society*, will be recalled. He was also the author of many other

articles, and of the well-known books: *Die Transformation der hyperelliptischen Functionen erster Ordnung nebst Anwendungen*, 1886; *Theorie der doppelt-periodischen Functionen einer veränderlichen Grösse*, 2 volumes, 1895–1897; and *Theorie der elliptischen Funktionen*, unter Mitwirkung von E. Naetsch, 1912.

Professors DUNHAM JACKSON, EDWARD KASNER, D. N. LEHMER, T. LEVI-CIVITA and H. L. RIETZ have been appointed associate editors of the *Bulletin of the American Mathematical Society* for 1921.

We welcome the first number of *Mathematics Teacher*, January, 1921, issued as the official organ of the National Council of Teachers of Mathematics, and devoted to the interests of mathematics in junior and senior high schools of the United States (compare *1920*, 474). It is edited by J. R. CLARK (editor-in-chief), E. R. SMITH (associate editor), J. A. FOBERG, (business manager), ALFRED DAVIS, H. D. GAYLORD, MARIE GUGLE, and J. W. YOUNG, with the cooperation of an advisory board consisting of W. H. METZLER (chairman), WILLIAM BETZ, W. E. BRECKENRIDGE, E. R. BRESLICH, J. C. BROWN, W. C. EELLS, G. W. EVANS, H. F. HART, W. W. HART, E. R. HEDRICK, H. M. KEAL, THEODORE LINDQUIST, W. A. LUBY, G. W. MYERS, J. H. MINNICK, W. D. REEVE, RALEIGH SCHORLING, H. E. SLAUGHT, D. E. SMITH, C. B. WALSH, and H. E. WEBB.

Professor FLORIAN CAJORI, professor of the history of mathematics at the University of California, addressed the Society of Sigma XI at Northwestern University, on December 13, 1920, on "Switzerland, the mother of American geodesy."

Dr. LUDWIK SILBERSTEIN, of the Eastman Kodak Company (*1920*, 474), delivered a series of six lectures at Cornell University, November 10 to Demceber 15, on the theory of relativity. An introductory lecture, on the experimental foundation for the theory, was given by Professors F. K. Richtmyer and E. R. Kennard on November 3.

We have referred elsewhere to addresses by Professors CAJORI, KARPINSKI, KELLOGG, and SMITH in joint sessions of the Mathematical Association of America, the American Mathematical Society and section "L" of the American Association for the Advancement of Science. The following papers in the history of science were also given on December 29–30, 1920, in separate sessions of Section L: "State of research in Egyptian science" by J. H. BREASTED, University of Chicago; "Early surveying and astronomical instruments in America" (illustrated) by Professor CAJORI, University of California; "Proposed periods in the history of astronomy in America" by W. C. RUFUS, University of Michigan; and "The history of physics" by Dr. H. A. BUMSTEAD, Yale University. (Dr. Bumstead died December 31, 1920).

In connection with the meeting of the American Mathematical Society in New York last December, the most notable event was the retirement of Professor F. N. COLE from his positions as secretary of the Society and as editor-in-chief

of the *Bulletin*. He served as secretary for twenty-five years, and as editor-in-chief for twenty-three years, with unfailing fidelity, tact, idealism, dignity and power. But he was much more than secretary and editor; the work which he carried on almost alone for years has now been distributed among four new officers of the Society.

The high position which American mathematics occupies in the world is due in no small measure to his quiet devotion, and sacrifice of leisure which, no doubt, he often coveted for continuing research investigations. Few of the younger mathematicians know of this work. His first paper, "The potential of a shell bounded by confocal ellipsoidal surfaces," was published in *Proceedings of the American Academy of Arts and Sciences*, in 1883. "A contribution to the theory of the general equation of the sixth degree" (his Harvard doctor's thesis), and a sixteen-page article, "Klein's Ikosaeder," appeared in *American Journal of Mathematics*, during the period 1885–87 that he was lecturer in mathematics at Harvard. This was just after his return from Germany "aglow with enthusiasm which Felix Klein inspired in his students," when he initiated at Harvard a new era[1] in graduate instruction in mathematics. While instructor and assistant professor at the University of Michigan, 1888–95, he published as follows: his English translation of Netto's *The Theory of Substitutions and its Applications to Algebra* (Ann Arbor, 1892); "The diurnal variation of barometric pressure," *Bulletin, U. S. Weather Bureau*, 1892; "On a simple group," *Mathematical Papers read at the International Mathematical Congress . . . 1893*, (New York, 1896); "The linear functions of a complex variable," *Annals of Mathematics*, 1890, 56 pages; "On rotations in space of four dimensions," *American Journal af Mathematics*, 1890; and a number of articles, one in collaboration with J. W. Glover, dealing with questions in the theory of groups. These included three in *American Journal of Mathematics*, two in *Bulletin of the New York Mathematical Society*, and two in *Quarterly Journal of Mathematics*.

He commenced to lay the foundation of the great monument of his life work shortly after his appointment in 1895 as professor of mathematics at Columbia University. His papers since that time have been: "On the factoring of large numbers" (where the often quoted factors of Mersenne's number, M_{67}, are given) in *Bulletin of the American Mathematical Society*, 1903; "The groups of order p^3q^β" in *Transactions of the American Mathematical Society*, 1904; and the memoir in collaboration with Professors White and Cummings, to which we have already referred (*1919*, 304), in *Memoirs of the National Academy of Sciences*, 1919. Some personal details regarding Professor Cole's life may be found in *National Cyclopædia of American Biography*, volume 13, 1906.

At the close of its sessions the Society testified to its appreciation of their retiring secretary and of his services, by the presentation of an address and a purse containing several hundred dollars.

Another event of the sessions, report on which will be of general interest, was the acceptance by the Council of the Society of the Bôcher Memorial Fund to

[1] For details the reader may, elsewhere in this MONTHLY (*1919*, 263), consult the report of a paper by Professor Osgood.

be held in trust, the income to be employed for the advancement of mathematical science. The council appointed a committee consisting of Professors W. F. OSGOOD (chairman), T. S. FISKE, C. N. HASKINS, O. D. KELLOGG, and E. B. VANVLECK to consider the most appropriate use to which the income of the Fund could be devoted.

The following four American doctorates in mathematics should be added to the lists already published (*1920*, 440 and *1921*, 44), making a total of twenty-two conferred in 1919–20: H. J. ETTLINGER, Harvard: "I. Existence theorems for the general real self-adjoint linear system of the second order. II. Oscillation theorems for the real self-adjoint linear system of the second order"; G. M. ROBISON, Cornell: "Divergent double sequences and series"; BIRD M. TURNER, Bryn Mawr: "Plane cubics with a given quadrangle of inflexions"; W. L. G. WILLIAMS, Chicago: "Fundamental systems of formal modular seminvariants of the binary cubic."

The *Bulletin of the American Mathematical Society* has published the following note (compare *1920*, 340): "Professor J. H. TANNER, of Cornell University, and Mrs. TANNER have given to the trustees of that institution fifty thousand dollars to establish a mathematical institute under the following stipulations. The money is to be allowed to accumulate without diversion for seventy-five years. At the end of that time one professor shall be appointed, whose duty it shall be to begin the formulation of plans for the proposed institute. At the end of each of the four succeeding periods of five years one or more additional professorships shall be established, the incumbents to collaborate in the same plans. The stipends of these professors shall be paid from the fund, but no other demands shall be made upon it until one hundred years from the date of the deed of gift (June, 1920), from which time the income of the entire sum shall be devoted to the maintenance of the institute; half of the expenditure of each year is to be applied to research in the mathematical sciences."

The National Research Council requested Professor H. L. RIETZ, of the University of Iowa, to call together a small group for a preliminary conference to discuss the desirability of a committee of the Council in the field of applications of mathematics to statistics, and to make appropriate recommendations. The group consisted of Professor J. W. GLOVER, Professor E. V. HUNTINGTON, Professor RAYMOND PEARL, and Professor W. M. PERSONS and the conference was held in Washington, January 21–22.

The Council of the American Association for the Advancement of Science chose Professor E. H. MOORE, of the University of Chicago, as president for the coming year. He will preside at Toronto in 1921 and deliver his address at Boston in 1922. *Science* refers to this "acknowledged leader of American mathematics": then adds: "It is now many years since that science which is fundamental to all others has supplied a president to the association, and it is fortunate that a representative could be selected with the unanimous approval of

all mathematicians." Professor W. D. CAIRNS and Professor G. A. MILLER were appointed members of the Council of the Association, and Executive Committee, respectively. In Section A, Professor OSWALD VEBLEN was elected vice-president and Professor W. H. ROEVER secretary for 1921.

We have already announced (*1921*, 44) the award by the Royal Society to Professor G. H. HARDY, of Oxford University, of a Royal Medal. Sir J. J. Thomson's statement in making the award is as follows (*Proceedings of the Royal Society*, series A, January 3, 1921):

"Prof. Hardy is well known both in this country and on the Continent for his researches in pure mathematics, particularly in the analytic theory of numbers and allied subjects. Immediately after taking his degree at Cambridge he engaged in a series of researches on the theory of functions of a real variable, from which results of the greatest importance and generality were obtained, at first by himself alone and later in collaboration with Mr. J. E. Littlewood. Perhaps his most remarkable work is his more recent work on Partitions, carried out in collaboration with the late Mr. S. Ramanujan, especially a paper on "Asymptotic Formulæ in Combinatory Analysis" (*Proc. Lond. Math. Soc.*), in which a system of approximations are obtained for $p(n)$, the number of unrestricted partitions of n and associated functions. The ingenuity of the method and the accuracy of the results are equally remarkable. For p (200) the authors' approximate formula gives the value 3,972,999,029,388.004, from which the true value differs by 0.004, an error of 1 in 10^{15}.

Among the more important researches of which Prof. Hardy is sole author may be mentioned papers on Dirichlet's Divisor Problem, on the representation of numbers as the sum of n squares, on the roots of the Riemann ζ-function, and on non-differentiable functions."

A similar award was made in 1919 to J. H. JEANS (*1920*, 46). Two Royal Medals are awarded each year. They were "instituted by George IV., and are awarded annually for the two most important contributions to science published in the British dominions not more than ten years nor less than one year from the date of the award."

There is a special call for copies of the MONTHLY for May, 1915. By an error in printing, the cover for that month read April, 1915, although the inside front page was marked correctly. Slips to be pasted over the incorrect date were sent out the next month, but a number of subscribers did not attend to this matter. As a result many libraries lack this one issue to complete their files. If anyone is willing to part with copies of the MONTHLY for May, 1915, such copies will be paid for by the Secretary of the Association at the rate of one dollar per copy, and will then be available for those desiring to complete their files. The same price will be paid for a limited number of copies for June and November, 1895, in volume 2.

THE JANUARY MEETING OF THE KANSAS SECTION.

The seventh regular meeting of the Kansas Section was held at the Central High School, Topeka, Kansas, on January 22, in conjunction with a meeting of the Kansas Association of Mathematics Teachers. Two sessions were held, the first of which was a joint session with the Kansas Association of Mathematics Teachers. Professor Shirk presided at the morning session and Professor Garrett at the afternoon session.

The attendance was fifty-two, including the following twenty-one members of the Association:

C. H. Ashton, Florence Black, R. H. Carpenter, Lucy Dougherty, W. H. Garrett, W. A. Harshbarger, T. B. Henry, Emma Hyde, T. Lindquist, Anna Marm, U. G. Mitchell, H. S. Myers, P. S. Pretz (institutional representative), B. L. Remick, D. H. Richert, J. A. G. Shirk, G. W. Smith, E. B. Stouffer, W. T. Stratton, J. J. Wheeler, A. E. White.

The following officers were elected for the coming year: Chairman, Professor SHIRK; Vice-Chairman, Miss HYDE; Secretary, Professor STOUFFER. It was voted to hold the next meeting at Topeka in January, 1922, in conjunction with the meeting of the Kansas Association of Mathematics Teachers.

The following eight papers were read:

(1) "Report of the Kansas committee coöperating with the National Committee on Mathematical Requirements" by Professor U. G. MITCHELL, University of Kansas;

(2) "Report on the preparation of college freshmen in entrance algebra" by Professor W. H. GARRETT, Baker University;

(3) Question box:

(*a*) "What high school teachers want to know about college mathematics" by Professor C. H. ASHTON, University of Kansas;

(*b*) "What college teachers want to know about high school mathematics" by Miss LUCY T. DOUGHERTY, High School, Kansas City;

(4) "Mathematics and statistics" by Professor W. T. STRATTON, Kansas State Agricultural College;

(5) "Hyperbolic functions" by Professor H. S. MYERS, Southwestern College;

(6) "A problem in calculus" by Professor A. E. WHITE, Kansas State Agricultural College;

(7) "Division of credit between college algebra, trigonometry, analytics and calculus" by Professor W. A. HARSHBARGER, Washburn College.

Most of the papers led to considerable discussion. Abstracts of the papers and discussions follow below, the numbers corresponding to numbers in list of titles.

1. The Kansas committee coöperating with the National Committee on Mathematical Requirements was first appointed in January, 1920, by the Kansas